UNIVERSITÄT KARLSRUHE
INSTITUT FÜR ANALYSIS
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2nd Exercise sheet

Functional Analysis

Deadline: Thursday, 15th Nov 2007, 15 p.m.

Exercise 5 (C)

Let X be a vector space equipped with a seminorm p. Let $M_X := \{x \in X : p(x) = 0\}$.

a) Show that M_X is closed, i.e. for any sequence $(x_n)_n \subseteq M_X$ and any $x \in X$ such that $p(x_n - x) \to 0$ it follows that $x \in M_X$.
In particular, the proposition about quotient spaces in the lecture can be applied to X, p and M_X : X/M_X is a normed space.

b) Let $n \in \mathbb{N}$, $\alpha \in (0,1]$ and Ω be a bounded and open subset of $\mathbb{R}^n$. Show that on

$$X := C^\alpha(\overline{\Omega}) = \{f : \overline{\Omega} \to \mathbb{R} : h_\alpha(f) := \sup_{\substack{x,y\in\overline{\Omega} \\ x\neq y}} \frac{|f(x) - f(y)|}{|x-y|^\alpha} < \infty\},$$

h_α is a seminorm.

c) Calculate M_X in this case.

d) Let $m \in \mathbb{N}$. Show that

$$C^{m,\alpha}(\overline{\Omega}) = \{f : \Omega \to \mathbb{R} : f \in C^m(\overline{\Omega}), \, h_\alpha(D^\beta f) < \infty \, \forall \beta \in \mathbb{N}^n : |\beta| = m\}$$

together with $\|f\|_{C^{m,\alpha}} := \sum_{|\beta|\leqslant m} \|D^\beta f\|_\infty + \sum_{|\beta|=m} h_\alpha(D^\beta f)$ is a normed space.
You can take for granted that $C^m(\overline{\Omega}) := \{f : \Omega \to \mathbb{R} : D^\beta f$ is bounded on Ω and can be continuously extended to $\overline{\Omega} \; \forall \beta : |\beta| \leqslant m\}$ is a normed space with $\|f\|_{C^m(\overline{\Omega})} := \sum_{|\beta|\leqslant m} \|D^\beta f\|_\infty$.

Exercise 6

Let X be a normed space and $T : X \to \mathbb{K}$ a linear mapping. Show that then

$$T \text{ is a bounded operator} \iff \ker T \text{ is closed.}$$

(Hint for „$\Leftarrow$“: look at the quotient mapping $\tilde{T} : X/\ker T \to \mathbb{K}$.)

— please turn over —

Exercise 7 (C)

Let $\mathbb{K}^{\mathbb{N}} = \{x = (x_k)_{k\in\mathbb{N}} : x_k \in \mathbb{K}\}$ and define

$$d(x, y) = \sum_{k=1}^{\infty} 2^{-k} \frac{|x_k - y_k|}{1 + |x_k - y_k|}, \qquad x = (x_k), y = (y_k) \in \mathbb{K}^{\mathbb{N}}.$$

a) Show that d is a metric on $\mathbb{K}^{\mathbb{N}}$.

b) For all sequences $\big(x^{(n)}\big)_{n\in\mathbb{N}} \subseteq \mathbb{K}^{\mathbb{N}}$ and all $x \in \mathbb{K}^{\mathbb{N}}$, we have that

$$d(x^{(n)}, x) \to 0 \text{ as } n \to \infty \iff \forall k \in \mathbb{N}: \; x_k^{(n)} \to x_k \text{ as } n \to \infty.$$

(Hint for a): For the triangle inequality, show first that the function $f : [0, \infty) \to \mathbb{R}$, $x \mapsto \frac{x}{1+x}$ satisfies $f(x + y) \leqslant f(x) + f(y)$ for $x, y \geqslant 0$.)

Exercise 8

Let (M, d) be a metric space and $V \subseteq M$. Show:

a) $\overline{V}$ is closed. V is closed if and only if $V = \overline{V}$.

b) V^o is open. V is open if and only if $V^o = V$.