UNIVERSITÄT KARLSRUHE
INSTITUT FÜR ANALYSIS
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3. Exercise sheet

Functional Analysis

Deadline: Thursday, 22th Nov 2007, 15 p.m.

Exercise 9 (C)

Let $X = C([0,1])$, equipped with its usual $\|\cdot\|_\infty$-norm. Prove or disprove the following assertions:

a) $A := \{f \in X : f([0,1]) = [0,1]\}$ is closed.

b) $B := \{f \in X : f \text{ is injective}\}$ is closed.

c) B is open.

d) $D := \{f \in X : f(\frac{1}{2}) = 0\}$ is closed.

Exercise 10

Let $(M, d_M), (N, d_N)$ be metric spaces and $f : M \to N$. Then the following are equivalent:

a) f is continuous on M.

b) For all open sets $U \subseteq N$, the set $f^{-1}(U)$ is open in M.

c) For all closed sets $A \subseteq N$, the set $f^{-1}(A)$ is closed in M.

Exercise 11 (C)

Let Ω be an open and bounded subset of $\mathbb{R}^n$ and $0 < \alpha \leqslant 1$. Prove that

a) $C^1(\overline{\Omega})$, equipped with the norm $\|f\|_{C^1(\overline{\Omega})} = \|f\|_\infty + \sum_{k=1}^n \|\partial_k f\|_\infty$ is complete.

b) The space of Hölder continuous functions $C^\alpha(\overline{\Omega})$, equipped with the norm $\|f\|_{C^\alpha(\overline{\Omega})} = \|f\|_\infty + \sup_{\substack{x \neq y \\ x,y \in \overline{\Omega}}} \frac{|f(x) - f(y)|}{|x-y|^\alpha}$ is complete.

Exercise 12

Let X be a Banach space. Prove that the set $\{J \in B(X) : \ J \text{ is an isomorphism}\}$ is open in $B(X)$.
Hint: For an isomorphism $J \in B(X)$ consider the open ball with radius $\frac{1}{\|J^{-1}\|}$.