UNIVERSITÄT KARLSRUHE
INSTITUT FÜR ANALYSIS
Prof.Dr. Lutz Weis
Christoph Kriegler

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4. Exercise sheet

Functional Analysis

Deadline: Thursday, 29th Nov 2007, 15 p.m.

Exercise 13 (C)

Let $(X, \|\cdot\|)$ be a normed space. Then X is complete if and only if for each sequence (x_n) in X with $\sum_{n=1}^{\infty} \|x_n\| < \infty$ there is an element $x \in X$ such that

$$\lim_{N\to\infty} \left\| x - \sum_{n=1}^{N} x_n \right\| = 0.$$

Exercise 14

Let M_1, M_2 be metric spaces, A a compact subset of M_1 and $f : M_1 \to M_2$ a continuous function. Show that

a) $f(A)$ compact in M_2,

b) if f is injective, then $f^{-1} : f(A) \to A$ is continuous.

Exercise 15

A subset S of ℓ^p $(1 \leqslant p < \infty)$ is relatively compact if and only if S is bounded and

$$\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall x \in S : \sum_{n=n_0}^{\infty} |x_n|^p < \varepsilon.$$

Exercise 16 (C)

Let $k \in C([0,1] \times [0,1])$. The operator $S : C([0,1]) \to C([0,1])$ given by

$$(Sf)(x) := \int_0^x k(x,y) f(y) dy.$$

is called *Volterra operator*. Show that S is compact.