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WS 2007/08
29th Nov 2007

5. Exercise sheet

Functional Analysis

Deadline: Thursday, 6th Dec 2007, 15 p.m.

On this exercise sheet, $\Omega \subseteq \mathbb{R}^n$ shall be L-measurable and μ denotes the Lebesgue measure on it. Further we assume $\mu(\Omega) > 0$.

Exercise 17 (C)

Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $A = (a_{jk}) \in \mathbb{K}^{\mathbb{N}\times\mathbb{N}}$ such that

$$\|A\|_{q,p} := \left(\sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} |a_{jk}|^q \right)^{p/q} \right)^{1/p} < \infty.$$

It was proved in the lecture that $(x_k)_{k\in\mathbb{N}} \mapsto (\sum_{k=1}^{\infty} a_{jk} x_k)_{j\in\mathbb{N}}$ defines a bounded linear operator $T \in B(\ell^p)$. Prove that T is compact.

Exercise 18 (C)

Let $m : \Omega \to \mathbb{K}$ be an L-measurable function and $1 \leqslant q \leqslant p \leqslant \infty$. For an L-measurable function $f : \Omega \to \mathbb{K}$ define $(Mf)(x) := m(x)f(x)$.

a) Prove that if $m \in L^r(\Omega, \mu)$ where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$, then $M \in B(L^p(\Omega, \mu), L^q(\Omega, \mu))$ and $\|M\| \leqslant \|m\|_r$.

b) Show that $M \in B(L^p(\Omega, \mu))$ if and only if $m \in L^\infty(\Omega, \mu)$ and that in this case $\|M\| = \|m\|_\infty$.

c) Assume now that $m \in L^\infty(\Omega, \mu)$. Determine the set of all $\lambda \in \mathbb{C}$ such that $\lambda - M$ is invertible in $B(L^p(\Omega, \mu))$.

Exercise 19

a) Suppose $\mu(\Omega) < \infty$ be bounded. and let $1 \leqslant q \leqslant p \leqslant \infty$. Then $L^p(\Omega, \mu) \subseteq L^q(\Omega, \mu)$ and, for $f \in L^p(\Omega, \mu)$,

$$\|f\|_q \leqslant \mu(\Omega)^{\frac{1}{q} - \frac{1}{p}} \|f\|_p.$$

b) If $1 \leqslant p, q \leqslant \infty$ with $p \neq q$, then neither $L^p(\mathbb{R}, \mu) \subseteq L^q(\mathbb{R}, \mu)$ nor $L^q(\mathbb{R}, \mu) \subseteq L^p(\mathbb{R}, \mu)$.

c) (Lyapunov's inequality) Let $1 \leqslant p_0, p_1 < \infty$ and $0 < \theta < 1$. Define p by $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Then $L^{p_0}(\Omega, \mu) \cap L^{p_1}(\Omega, \mu) \subseteq L^p(\Omega, \mu)$ and, for $f \in L^{p_0}(\Omega, \mu) \cap L^{p_1}(\Omega, \mu)$,

$$\|f\|_p \leqslant \|f\|_{p_0}^{1-\theta} \|f\|_{p_1}^{\theta}.$$

Hint: Hölder's inequality.

Exercise 20

Let $(f_n)_{n \in \mathbb{N}} \subseteq \mathcal{L}^1(\Omega, \mu)$ such that $\int_\Omega |f_n(\omega)| d\omega \to 0$.

a) Show that there exists a subsequence $(f_{n_k})_k$ and a null set $N \subseteq \Omega$ such that $f_{n_k}(\omega) \to 0$ for all $\omega \in \Omega \backslash N$.

b) Show that in general, without passing to the subsequence, $f_n(\omega)$ doesn't need to be convergent in any point.

Principal theorems on L-integrable functions

Let $g, f_1, f_2, \ldots$ be L-measurable functions $\Omega \to \mathbb{K}$.

- Definition: An assertion depending on $\omega \in \Omega$ is said to hold for **almost all** (**a.a.** in short) ω if there is a set of measure zero (also called a **null set**) N such that for all $\omega \in \Omega \backslash N$, the assertion holds. One also says that the assertions holds **almost everywhere** (**a.e.**).

- **(Monotone convergence)** If $0 \leqslant f_1(\omega) \leqslant f_2(\omega) \leqslant \ldots$ for a.a. ω and there is a $C > 0$ such that $\int_\Omega f_k d\mu \leqslant C$, then $f_k(\omega)$ converges for a.a. ω and setting $f(\omega) := \lim_k f_k(\omega)$ where this limit exists and for example $f(\omega) := 0$ elsewhere, we have that: f is L-integrable and $\int_\Omega f d\mu = \lim_k \int_\Omega f_k d\mu$.

- **(Dominated convergence)** Let g be L-integrable, $f = \lim_k f_k$ exist a.e. and $|f_k| \leqslant g$ a.e. Then f (and also f_k) is L-integrable, $\int_\Omega f d\mu = \lim_k \int_\Omega f_k d\mu$ and also $\lim_k \int_\Omega |f - f_k| d\mu = 0$.

- Let $\Omega_i \subseteq \mathbb{R}^{n_i}$ be L-measurable for $i = 1, 2$ and $f : \Omega_1 \times \Omega_2 \to \mathbb{K}$ be L-measurable. Denote μ the Lebesgue measure on $\Omega_1 \times \Omega_2 \subseteq \mathbb{R}^{n_1+n_2}$ and μ_i the Lebesgue measure on Ω_i $(i = 1, 2)$.

  a) **(Tonelli)** If $\displaystyle\int_{\Omega_1} \left(\int_{\Omega_2} |f(s,t)| d\mu_2(t) \right) d\mu_1(s) < \infty$, then f is L-integrable.

  b) **(Fubini)** If f is L-integrable, then $s \mapsto f(s,t)$ is L-integrable for a.a. $t \in \Omega_1$ and the a.e. defined function $t \mapsto \int_{\Omega_2} f(s,t) ds$ is L-integrable with $\int_{\Omega_1 \times \Omega_2} f(s,t) d\mu(s,t) = \int_{\Omega_1} \left(\int_{\Omega_2} f(s,t) d\mu_2(t) \right) d\mu_1(s)$.

- **(Hölder)** Let $1 \leqslant p, q, r \leqslant \infty$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Then for $f \in L^p(\Omega, \mu)$ and $g \in L^q(\Omega, \mu)$, the pointwise product fg is in $L^r(\Omega, \mu)$ and $\|fg\|_r \leqslant \|f\|_p \|g\|_q$.