UNIVERSITÄT KARLSRUHE
INSTITUT FÜR ANALYSIS
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6. Exercise sheet

Functional Analysis

Deadline: Thursday, 13th Dec 2007, 15 p.m.

Exercise 21

(Conditional expectation operator) Let $\Omega \subseteq \mathbb{R}^n$ be an L-measurable set with $\mu(\Omega) \in (0, \infty)$. Let further $m \in \mathbb{N}$ and $\mathcal{A} = \{A_1, \ldots, A_m\}$ with A_j pairwise disjoint L-measurable subsets of Ω with positive measure s. th. $\bigcup_{j=1}^m A_j = \Omega$. Define for $f \in L^1(\Omega)$

$$E_{\mathcal{A}} f(s) := \sum_{j=1}^m \frac{1}{\mu(A_j)} \int_{A_j} f(t) dt \, \chi_{A_j}(s).$$

a) Show that $E_{\mathcal{A}}$ is of the form $E_{\mathcal{A}} f(u) = \int_\Omega k(u, v) f(v) dv$, $f \in L^1(\Omega)$ for some $k : \Omega \times \Omega \to \mathbb{R}$.

b) Deduce that $\|E_{\mathcal{A}}\|_{B(L^p(\Omega))} \leqslant 1$ for all $1 \leqslant p \leqslant \infty$.

c) Let Ω be in addition compact. Let further $\mathcal{A}_k = \{A_{1k}, \ldots, A_{m_k k}\}$ be a sequence of partitions with the above properties. Assume that $d_k := \sup\{|t - s| : t, s \in A_{jk}, j = 1, \ldots, m_k\} \to 0$ for $k \to \infty$. Show that for $1 \leqslant p < \infty$ and $f \in L^p(\Omega)$

$$\|E_{\mathcal{A}_k} f - f\|_p \longrightarrow 0.$$

Exercise 22 (C)

(a) Let $1 \leqslant p < \infty$. For a subset $K \subseteq L^p(\mathbb{R}^n, \mu)$ we consider the following three conditions:

(C1) K is bounded.

(C2) $\sup_{f \in K} \|f(\cdot + h) - f(\cdot)\|_p \to 0$ for $|h| \to 0$.

(C3) $\sup_{f \in K} \int_{\mathbb{R}^n \setminus B(0,R)} |f(t)|^p dt \to 0$ for $R \to \infty$.

(So, (C1) and (C3) are in analogy to exercise 15, (C2) is new.)

Show that if K is relatively compact, then (C1),(C2) and (C3) are true.

(b) Let $p = \infty$. Is $\|f(\cdot + h) - f\|_p \to 0$ as $h \to 0$ still true?

Exercise 23

The goal of this exercise is to show the reverse implication of ex. 22 a): For $1 \leqslant p < \infty$, let $K \subseteq L^p(\mathbb{R}^n, \mu)$ satisfy (C1), (C2) and (C3). Then K is relatively compact.

So let $1 \leqslant p < \infty$ and $K \subseteq L^p(\mathbb{R}^n, \mu)$ satisfy (C1), (C2) and (C3). Let $\varphi : \mathbb{R}^n \to [0, \infty)$ be Lipschitz continuous, such that $\varphi(x) = 0$ for $|x| \geqslant 1$ and $\int_{B(0,1)} \varphi(x)dx = 1$. Let further for $\varepsilon > 0$, $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(\frac{x}{\varepsilon})$. Define for ε, R and $f \in K$

$$f_{\varepsilon,R} := \int_{B(0,R)} \varphi_\varepsilon(x-y)f(y)dy.$$

From the lecture, we know:

- $|f_{\varepsilon,R}(x) - f(x)| \leqslant \int_{\mathbb{R}^n} \varphi_\varepsilon(y)|f(x-y) - f(x)|dy + |f(x)|\chi_{\mathbb{R}^n \setminus B(0,R-\varepsilon)}$.
- $\| \int_{\mathbb{R}^n} \varphi_\varepsilon(y)|f(\cdot - y) - f(\cdot)|dy\|_p \leqslant \sup_{y \in B(0,\varepsilon)} \|f(\cdot - y) - f\|_p$.
- So by (C2) and (C3), $\sup_{f\in K} \|f_{\varepsilon,R} - f\|_p \to 0$ as $\varepsilon \to 0$ and $R \to \infty$.

a) Show that for all $\varepsilon, R > 0$, $K_{\varepsilon,R} := \{f_{\varepsilon,R} : f \in K\}$ is a bounded and equicontinuous subset of $C(B(0, R+\varepsilon))$.

b) Show that for all $\varrho > 0$, there exist $\varepsilon, R > 0$, $m \in \mathbb{N}$ and $g_1, \ldots, g_m \in C(B(0, R + \varepsilon)) \subseteq L^p(\mathbb{R}^n)$ such that $K \subseteq \bigcup_{j=1}^m B(g_j, \varrho)$. (The balls $B(g_j, \varrho)$ are with respect to $L^p(\mathbb{R}^n)$.)

c) Conclude that K is relatively compact.

Exercise 24 (C)

Let $1 \leqslant p < q < \infty$ and $\Omega \subseteq \mathbb{R}^n$ be L-measurable. Let further $D \subseteq L^1(\Omega, \mu) \cap L^\infty(\Omega, \mu)$. In particular, $D \subseteq L^r(\Omega, \mu)$ for all $r \in (1, \infty)$ by the Lyapunov inequality, ex. 19c). We assume that

- D is dense in $L^q(\Omega, \mu)$.
- $D \cdot D \subseteq D$, i.e. for all $f, g \in D : \ f \cdot g \in D$.

Show that D is dense in $L^p(\Omega, \mu)$.
Hint: Hölder inequality, as on the bottom of the back of exercise sheet 5.