UNIVERSITÄT KARLSRUHE
INSTITUT FÜR ANALYSIS
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7. Exercise sheet

Functional Analysis

Deadline: Thursday, 20th Dec 2007, 15 p.m.

Exercise 25

(Integration by parts)

a) Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable and have compact support. Then

$$\int_{\mathbb{R}^n} \frac{\partial f}{\partial x_j}(x)dx = 0 \quad \forall j = 1, \ldots, n.$$

(Hint: Fubini's theorem)

b) Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable. f shall have compact support. Then

$$\int_{\mathbb{R}^n} \frac{\partial f}{\partial x_j}(x)g(x)dx = -\int_{\mathbb{R}^n} f(x)\frac{\partial g}{\partial x_j}(x)dx \quad \forall j = 1, \ldots, n.$$

c) Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $f, g : \Omega \to \mathbb{R}$ be m-times continuously differentiable for some $m \in \mathbb{N}$. Further one of the functions shall have compact support. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi-index with length $|\alpha| \leqslant m$. Then

$$\int_\Omega D^\alpha f(x)g(x)dx = (-1)^{|\alpha|} \int_\Omega f(x)D^\alpha g(x)dx.$$

Hint: Find an appropriate continuation of fg on $\mathbb{R}^n$.

Exercise 26 (C)

Let $I = (0, 2)$ and $f, g, h : I \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} x, & 0 < x \leqslant 1, \\ 1, & 1 < x < 2, \end{cases} \qquad g(x) := \begin{cases} 1, & 0 < x \leqslant 1, \\ 0, & 1 < x < 2, \end{cases} \qquad h(x) := \begin{cases} x, & 0 < x \leqslant 1, \\ 2, & 1 < x < 2. \end{cases}$$

Show that for $1 \leqslant p \leqslant \infty$

a) $f \in H^{1,p}(I)$ and $f' = g$ in the weak sense.

b) $h \notin H^{1,p}(I)$.

— please turn over —

Exercise 27

Let $I = (a, b)$, $-\infty \leqslant a < b \leqslant \infty$, be an open interval.

a) Let $f \in L^p(I)$. Prove that if

$$\int_a^b f\varphi' d\mu = 0 \qquad \text{for all } \varphi \in C_c^\infty(I),$$

then there is a constant c such that $f = c$ a.e.

b) Let $u \in H^{1,p}(I)$. Then there is a function $\tilde{u} \in C(\bar{I})$ such that $u = \tilde{u}$ a.e. and

$$\tilde{u}(y) - \tilde{u}(x) = \int_x^y u'(t)dt \qquad \forall x, y \in I.$$

Exercise 28 (C)

(Product rule for weak derivatives)
Let $\Omega \subseteq \mathbb{R}^n$ be open and $1 \leqslant p, q \leqslant \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Prove that:

$$f \in H^{1,p}(\Omega) \text{ and } g \in H^{1,q}(\Omega) \Longrightarrow f \cdot g \in H^{1,1}(\Omega) \text{ and } (f \cdot g)' = f'g + fg'.$$

Hint: For $1 \leqslant p < \infty$, $H^{1,p}(\Omega) \cap C^\infty(\Omega)$ is dense in $H^{1,p}(\Omega)$.