UNIVERSITÄT KARLSRUHE
INSTITUT FÜR ANALYSIS
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**8. Exercise sheet**

**Functional Analysis**

Deadline: Thursday, 10th Jan 2008, 15 p.m.

### Exercise 29 (C)

(a) $\mathbb{Q}$ cannot be written as a countable intersection of open subsets in $\mathbb{R}$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then the set $\{t \in \mathbb{R} : f \text{ is continuous in } t\}$ can be written as a countable intersection of open subsets in $\mathbb{R}$.

(Hint: You can try to combine sets of the form
$A_{n,m} = \left\{ t \in \mathbb{R} : \sup\{|f(x) - f(y)| : |x - t|, |y - t| < \frac{1}{n}\} < \frac{1}{m} \right\}$.)

(c) There is no function $f : \mathbb{R} \to \mathbb{R}$ that is continuous in $\mathbb{Q}$, but not continuous in $\mathbb{R} \setminus \mathbb{Q}$.

### Exercise 30 (C)

(a) Let $X, Y, Z$ be normed spaces and $T : X \times Y \to Z$ bilinear. Show that $T$ is continuous if and only if there is $M > 0$ such that for all $x \in X$ and all $y \in Y$

$$\|T(x, y)\|_Z \leqslant M \|x\|_X \|y\|_Y.$$

(b) Let $X, Y$ be Banach spaces and $T : X \times Y \to \mathbb{K}$ bilinear. Show that if $T$ is separately continuous (i.e., for all $x \in X$, $y \mapsto T(x, y)$ is continuous and, for all $y \in Y$, $x \mapsto T(x, y)$ is continuous), then $T$ is continuous.

### Exercise 31

Let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $x = (x_k)$ is a sequence in $\mathbb{K}$ such that for every $y = (y_k) \in \ell^p$ the series $\sum_{k=1}^{\infty} x_k y_k$ converges, then $x \in \ell^q$.

**Exercise 32**

Prove theorem 12.13 from the lecture:
Let $X$ be a Banach space, $M \subseteq X$ a closed subspace. For $N \subseteq X$ a further closed subspace put $M \oplus_1 N := M \times N$ equipped with the norm $\|(x, y)\|_1 := \|x\| + \|y\|$. Then the following conditions are equivalent:

**a)** There exists a continuous projection $P \in B(X)$ such that $P(X) = M$.

**b)** There exists a closed subspace $N \subseteq X$ with $X = M \oplus N$.

**c)** There exists a closed subspace $N \subseteq X$ such that $J : \begin{cases} M \oplus_1 N & \longrightarrow X \\ (x, y) & \mapsto x + y \end{cases}$ is a (surjective) isomorphism, i.e. $\forall\, z \in X\, \exists\, x \in M, y \in N : x+y = z$ and $\exists C > 0 : \forall x \in M,\, y \in N : \frac{1}{C}(\|x\| + \|y\|) \leqslant \|x+y\| \leqslant \|x\| + \|y\|$.