UNIVERSITÄT KARLSRUHE
INSTITUT FÜR ANALYSIS
Prof.Dr. Lutz Weis
Christoph Kriegler

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9. Exercise sheet

Functional Analysis

Deadline: Thursday, 17th Jan 2008, 15 p.m.

Exercise 33 (C)

Which of the following operators A_j are closed? Give a proof for your answer.

(a) $D_1 := \{(x_k) \in \ell_2 : \ (kx_k) \in \ell_1\}$ with $\|\cdot\|_2$,
$A_1 : D_1 \to \ell_2$, $(x_k) \mapsto (kx_k)$.

(b) $D_2 := \{(x_k) \in \ell_2 : \ (kx_k) \in \ell_2\}$ with $\|\cdot\|_2$,
$A_2 : D_2 \to \ell_2$, $(x_k) \mapsto (kx_k)$.

(c) $D_3 := \{f \in C^2[0,1] : f(0) = f'(0) = 0\}$ with $\|\cdot\|_\infty$,
$A_3 : D_3 \to C[0,1]$, $A_3 f := f''$.

(d) $D_4 := \{f \in C[0,1] \cap C^1(0,1] : \ \exists h \in C[0,1]$ with $h(x) = xf'(x)$ for $x \in (0,1]\}$ with $\|\cdot\|_\infty$,
$A_4 : D_4 \to C[0,1]$, $A_4 f := h$.

Exercise 34

Let X, Y be Banach spaces and $D(A) \subseteq X$ be a linear subspace.
Suppose that $A : D(A) \to X$ is a closed operator.

a) Let $T : X \to X$ be linear, $J : X \to Y$ linear, injective and continuous, and let $JT : X \to Y$ be continuous. Show that then T is also continuous.
Hint: Closed graph theorem.

b) If $T \in B(X)$ such that $T(X) \subseteq D(A)$, then $AT \in B(X)$.

c) If $T \in B(X)$ then $TA : X \supseteq D(A) \to X$ is not necessarily closed.

d) Kern A is a closed subspace of X.

Exercise 35

Let $X = C[0,1]$ and $V \in B(X)$ defined by $Vf(x) := \int_0^x f(t)dt$. Prove that $\sigma(V) = \{0\}$ and $\sigma_p(V) = \emptyset$.
$\sigma_p(V)$ stands for the **point spectrum** of V : $\sigma_p(V) = \{\lambda \in \mathbb{C} : \ \lambda - V \text{ not injective}\} \subseteq \sigma(V)$.

Exercise 36 (C)

Let X be a Banach space and $T \in B(X)$. Prove that $\sigma(T^n) = [\sigma(T)]^n$ for all $n \in \mathbb{N}$.