UNIVERSITÄT KARLSRUHE
INSTITUT FÜR ANALYSIS
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10. Exercise sheet

Functional Analysis

Deadline: Thursday, 24th Jan 2008, 15 p.m.

Exercise 37

Let A be a subset of a Hilbert space H. Prove that

a) if $B \subseteq A$ then $A^\perp \subseteq B^\perp$.

b) $A^\perp$ is a closed linear subspace of H.

c) $A \subseteq (A^\perp)^\perp$, $A^\perp = (\overline{\text{span} A})^\perp$, $((A^\perp)^\perp)^\perp = A^\perp$.

Exercise 38 (C)

Let H be a Hilbert space, Y a linear subspace of H, and $x, y \in H$. Prove that

a) x, y are orthogonal if and only if $\|x + \alpha y\| = \|x - \alpha y\|$ for all $\alpha \in \mathbb{K}$.

b) $x \in Y^\perp$ if and only if $\|x - y\| \geqslant \|x\|$ for all $y \in Y$.

Exercise 39 (C)

a) For $n \in \mathbb{N}$, the Rademacher functions r_n are defined by $r_n(t) = \text{sign} \sin(2^n \pi t)$. Show that $\{r_n : n \in \mathbb{N}\}$ is an orthonormal system in $L_2([0, 1])$. Is $\{r_n : n \in \mathbb{N}\}$ an orthonormal basis of $L_2([0, 1])$?

b) Let $\varphi = \chi_{(0,\frac{1}{2})} - \chi_{(\frac{1}{2},1)}$. For $n = 2^k + j$ ($k \in \mathbb{N}_0$, $j = 0, \ldots, 2^k - 1$), define the Haar functions $h_n : [0, 1] \to \mathbb{R}$ by $h_n(t) = 2^{k/2}\varphi(2^k t - j)$; moreover let $h_0(t) = \chi_{(0,1)}$. Prove that $\{h_n : n \in \mathbb{N}_0\}$ is an orthonormal basis of $L_2([0, 1])$.

Hint for (b): Show first that $\{h_n : n \in \mathbb{N}_0\}$ is an orthonormal system in $L_2([0, 1])$. Then prove that $f \mapsto \sum_{n=0}^{2^m-1} \langle f, h_n \rangle h_n$ is the orthogonal projection on the subspace of functions that are constant on the dyadic intervals $(l2^{-m}, (l + 1)2^{-m})$, $l = 0, \ldots 2^m - 1$, in $L_2([0, 1])$.

— please turn over —

Exercise 40

Let X be a Hilbert space and $0 \neq P \in B(X)$ a projection. Prove that the following assertions are equivalent:

a) P is an orthogonal projection.

b) $\|P\| = 1$.

c) P is symmetric, i.e., $\langle Px, y\rangle = \langle x, Py\rangle$ for all $x, y \in X$.