UNIVERSITÄT KARLSRUHE
INSTITUT FÜR ANALYSIS
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11. Exercise sheet

Functional Analysis

Deadline: Thursday, 31st Jan 2008, 15 p.m.

Exercise 41 (C)

Let H be a real Hilbert space, let $a : H \times H \to \mathbb{R}$ be bilinear, continuous, symmetric and coercive, and let $\varphi \in H'$. Define the (nonlinear) functional $F : H \to \mathbb{R}$ by

$$F(v) = \frac{1}{2}a(v,v) - \varphi(v), \qquad v \in H.$$

a) Show that there is a unique $u \in H$ such that $a(u,v) = \varphi(v)$ for all $v \in H$.

b) Prove that u is given by the absolute minimum of F, i.e., $F(v) > F(u)$ for all $v \in H$ with $v \neq u$.

Exercise 42

We consider the Neumann problem

$$(\mathrm{NP}) \begin{cases} -u'' + u &= f \qquad \text{on } I = (0,1) \\ \quad u'(0) &= \alpha \\ \quad u'(1) &= \beta \end{cases}$$

where $\alpha, \beta \in \mathbb{R}$ and $f : [0,1] \to \mathbb{R}$ continuous. Prove that the following assertions hold.

a) If there is $u \in C^2[0,1]$ such that (NP) is satisfied, then for all $v \in H^{1,2}(I)$,

$$(\mathrm{WNP}) \qquad \int_0^1 u'v' d\mu + \int_0^1 uv\, d\mu = \int_0^1 fv\, d\mu - \alpha v(0) + \beta v(1).$$

b) The linear functional $\varphi : H^{1,2}(I) \to \mathbb{R}$ given by $\varphi(v) := \int_0^1 fv\, d\mu - \alpha v(0) + \beta v(1)$ is continuous.

c) There exists a unique function $u \in H^{1,2}(I)$ such that (WNP) is satisfied. (Use the Riesz Representation Theorem.)

d) The function u from (c) is in $C^2[0,1]$ and satisfies (NP).

— please turn over —

Exercise 43 (C)

Let X be a Pre-Hilbert space.

a) For a sequence $(x_n) \subseteq X$ and $x \in X$ prove that

$$\|x_n - x\| \to 0 \qquad \Leftrightarrow \qquad \begin{cases} \|x_n\| \to \|x\| \text{ and} \\ \langle x_n, y\rangle \to \langle x, y\rangle \text{ for all } y \in Y. \end{cases}$$

b) Show that X is complete if and only if the Riesz Representation Theorem holds in X, i.e., for each functional $\varphi \in X'$ there is $x \in X$ such that $\varphi(y) = \langle x, y\rangle$ for all $y \in X$.

Exercise 44

Let $H = L^2([0,1])$ and $k, h \in L^2([0,1]^2)$. Define T and $S \in B(H)$ by $(Tf)(x) := \int_0^1 k(x,y)f(y)dy$ and $(Sf)(x) := \int_0^1 h(x,y)f(y)dy$.

a) Determine TS in terms of k and h.

b) Determine T^* in terms of k.

c) Let now $T \in B(L^2([0,1]))$, $(Tf)(x) = \int_0^x f(y)dy$.
Do we have $TT^* = T^*T$, i.e., is T a normal operator?