UNIVERSITÄT KARLSRUHE
INSTITUT FÜR ANALYSIS
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WS 2007/08
31st Jan 2008

12. Exercise sheet

Functional Analysis

Deadline: Thursday, 7th Feb 2008, 15 p.m.

Exercise 45

Let X, Y be Hilbert spaces and $K \in B(X, Y)$. Prove that the following assertions are equivalent.

(i) K is compact.
(ii) For all orthonormal systems $\{e_k\}_{k=1}^{\infty} \subseteq X$, we have that $\|Ke_k\|_Y \to 0$.
(iii) For all orthonormal systems $\{e_k\}_{k=1}^{\infty} \subseteq X$ and $\{f_k\}_{k=1}^{\infty} \subseteq Y$, $\langle Ke_k, f_k \rangle \to 0$.

Exercise 46

Let H be a complex Hilbert space and $T \in B(H)$.
(a) T is self-adjoint if and only if $\langle Tx, x \rangle \in \mathbb{R}$ for all $x \in H$.
(b) If T is self-adjoint, then $\|T\| = \sup_{\|x\| \leqslant 1} |\langle Tx, x \rangle|$.

Exercise 47 (C)

The von Neumann Ergodic Theorem.
Let H be a Hilbert space and let $T \in B(H)$ be unitary. Define $M := \{x \in H : Tx = x\}$, which is a closed subspace of H. The goal of the exercise is to show that for all $x \in H$

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} T^n x$$

converges against Px, the orthogonal projection onto M.

- Show the assertion in the case $x \in \operatorname{Ran} P$.
- Show that it suffices to prove the assertion for $x \in \operatorname{Ran} P$ and $x \in \operatorname{Kern} P$.
- Show that $\operatorname{Kern} P = \overline{\operatorname{Ran}(I - T)}$.
- Show the assertion for $x \in \operatorname{Ran}(I - T)$.
- Conclude.

Exercise 48 (C)

Let $H := L^2[0,1]$ and $V \in B(H)$ be defined by

$$V(f)(x) = \int_0^x f(t)dt.$$

We further put $R = VV^*$, which is a compact and selfadjoint operator. (Why?)

a) Show that $R(f)(x) = \int_0^x tf(t)dt + x\int_x^1 f(t)dt$.

b) Show that the eigenvalues of R are given by $\alpha_n = \frac{4}{\pi^2}\frac{1}{(2n-1)^2}$, $n \in \mathbb{N}$, that the corresponding eigenspaces are 1-dimensional and spanned by $\varepsilon_n(x) = \sqrt{2}\sin\left((2n-1)\frac{\pi x}{2}\right)$.

c) Deduce that the ε_n form an orthonormal basis of H.