Problem 45

Let X, Y be Hilbert spaces and $K \in B(X, Y)$. Prove that the following assertions are equivalent.

(i) K is compact.

(ii) For all orthonormal systems $\{e_k\}_{k=1}^{\infty} \subseteq X$, we have that $\|Ke_k\|_Y \to 0$.

(iii) For all orthonormal systems $\{e_k\}_{k=1}^{\infty} \subseteq X$ and $\{f_k\}_{k=1}^{\infty} \subseteq Y$, $\langle Ke_k, f_k\rangle \to 0$.

Solution 45

Remark: Since K is bounded, we have that for all $y \in Y$, $\langle Ke_k, y\rangle_Y = \langle e_k, K^*y\rangle_X \to 0$. So $Ke_k \to 0$ weakly.

(i) $\Rightarrow$ (ii): Assume that $(\|Ke_k\|)_k$ does not converge to 0. Then there is $c > 0$ and a subsequence $(e_{k_n})_n$ of (e_k) such that $\|Ke_{k_n}\| \geq c$ for all $n \in \mathbb{N}$. Since K is compact and (e_k) is a bounded sequence, we can choose the subsequence $(e_{k_n})_n$ such that (Ke_{k_n}) converges with respect to the norm in Y. Because of $Ke_{k_n} \to 0$ weakly, the limit of (Ke_{k_n}) has to be 0. But this contradicts our assumption that $\|Ke_{k_n}\| \geq c$ for all $n \in \mathbb{N}$.

(ii) $\Rightarrow$ (i): Take $e_1 \in X$ such that $\|e_1\| = 1$ and $\|Ke_1\| \geq \frac{1}{2}\|K\|$. Inductively choose $e_{n+1} \in (\operatorname{span}\{e_1, \dots, e_k\})^{\perp}$ with $\|e_{n+1}\| = 1$ and $\|Ke_{n+1}\| \geq \frac{1}{2}\|K(I - P_n)\|$, where P_n is the orthogonal projection on $\operatorname{span}\{e_1, \dots, e_n\}$. Then $K_n = KP_n \in B(X, Y)$ are finite dimensional and by assumption

$$\|K - K_n\| = \|K(I - P_n)\| \leq 2\|Ke_{n+1}\| \to 0$$

for $n \to \infty$. So K is compact (cf. Corollary 7.6).

(ii) $\Rightarrow$ (iii) is clear.

(iii) $\Rightarrow$ (ii): Assume that there is a orthonormal sequence (e_k) in X such that (Ke_k) does not converge to 0. Without loss of generality we can assume that there is $c > 0$ with $\|Ke_k\| \geq c$ for all $k \in \mathbb{N}$ (take a subsequence if necessary).
Let $f_1 = \frac{Ke_1}{\|Ke_1\|}$ and denote by P_1 the orthogonal projection onto $\operatorname{span}\{f_1\}$. Since $Ke_k \to 0$ weakly, $\langle Ke_k, f_1\rangle \to 0$. Hence there is $k_2 \in \{2, 3, \dots\}$ such that

$$\|P_1(Ke_{k_2})\| = |\langle Ke_{k_2}, f_1\rangle| \leq \frac{c}{\sqrt{2}}.$$

Write $y_2 := Ke_{k_2}$. Then

$$\|(I - P_1)y_2\|^2 = \|y_2\|^2 - \|P_1 y_2\|^2 \geq c^2 - \frac{c^2}{2} = \frac{c^2}{2}.$$

Now we choose $f_2 = \frac{(I-P_1)y_2}{\|(I-P_1)y_2\|}$. Then $|\langle y_2, f_2\rangle| = \|(I - P_1)y_2\| \geq \frac{c}{\sqrt{2}}$. By P_2 we denote the orthognonal projection onto $\operatorname{span}\{f_1, f_2\}$.
We continue inductively: Assume we already have defined $f_1, \dots, f_n$. By P_n we denote the orthognonal projection onto $\operatorname{span}\{f_1, \dots, f_n\}\}$. Then, since $Ke_k \to 0$ weakly, $\langle Ke_k, f_j\rangle \to 0$ for $j = 1, \dots, n$. Hence there is $k_{n+1} \in \{k_n + 1, k_n + 2, \dots\}$ such that

$$\|P_n(Ke_{k_{n+1}})\|^2 = \sum_{j=1}^{n} |\langle Ke_{k_{n+1}}, f_j\rangle|^2 \leq \frac{c^2}{2}.$$

Write $y_{n+1} := Ke_{k_{n+1}}$. Then

$$\|(I - P_n)y_{n+1}\|^2 = \|y_{n+1}\|^2 - \|P_n y_{n+1}\|^2 \geq c^2 - \frac{c^2}{2} = \frac{c^2}{2}.$$

Now we choose $f_{n+1} = \frac{(I-P_n)y_{n+1}}{\|(I-P_n)y_{n+1}\|}$. Then $|\langle y_{n+1}, f_{n+1}\rangle| = \|(I - P_n)y_{n+1}\| \geq \frac{c}{\sqrt{2}}$. So we have seen that for the constructed orthonormal sequence (f_k) in Y we have that $|\langle Ke_{k_n}, f_n\rangle| \geq \frac{c}{\sqrt{2}}$ for all $n \geq 2$. But this contradicts our assumption. $\square$

Exercise 48 Let $H := L^2([0,1])$ and $V \in \mathcal{B}(H)$

defined by

$$Vf(x) := \int_0^x f(y)\,dy \qquad \left(= \int_0^1 \mathbb{1}_{[0,x]}(y) f(y)\,dy \right)$$

We further put $R := W^*$.

Before starting the exercise as given on the ex. sheet, we will collect some facts about the operator V.

(1) For all $f \in H$, Vf is a <u>continous function</u>.

<u>Proof</u>: Let $x \in [0,1]$. Then (with $\langle a,b\rangle := [\min\{a,b\}, \max\{a,b\}]$)

$$|Vf(x+h) - Vf(x)| = \left| \int_x^{x+h} f(y)\,dy \right| = \int_0^1 \mathbb{1}_{[x,x+h]}(y) f(y)\,dy.$$

We have $\mathbb{1}_{[x,x+h]}(y) f(y) \xrightarrow{h\to 0} \begin{cases} 0 & \text{for } y \neq x \\ f(x) & \text{for } y \neq x \end{cases}$,

hence $\mathbb{1}_{[x,x+h]} \cdot f \longrightarrow 0$ pointwise a.e. for $h \to 0$,

and $|\mathbb{1}_{[x,x+h]} \cdot f| \leq |f| \in L^2([0,1]) \hookrightarrow L^1([0,1])$.

So Lebesgue's theorem yields

$$\int_0^1 \mathbb{1}_{[x,x+h]}(y) f(y)\,dy \xrightarrow{h\to 0} 0.$$

(2) For all $f \in H$, Vf is even <u>weakly differentiable</u> with <u>weak derivative</u> f.

<u>Proof</u>. Let $\varphi \in C_c^\infty((0,1))$, then we have:

$$-\int_0^1 Vf(x)\,\varphi'(x)\,dx = -\int_0^1 \int_0^1 \mathbb{1}_{[0,x]}(y) f(y) \varphi'(x)\,dy\,dx$$

<u>Fubini</u>: $f \otimes \varphi' \in \mathcal{L}^1$

$$= -\int_0^1 \left[\int_0^1 \mathbb{1}_{[0,x]}(y)\, \varphi'(x)\,dx \right] f(y)\,dy$$

$$= \int_0^1 \left(-\int_y^1 \varphi'(x)\,dx \right) f(y)\,dy$$

$$= \int_0^1 \Big(\underbrace{-\varphi(1)}_{=0} + \varphi(y) \Big) f(y)\,dy = \int_0^1 \varphi(y) f(y)\,dy.$$

By definition, it follows that f is weakly db. with weak derivative $(Vf)' = f$. //

<u>Remark</u>. In fact, more is true: the function Vf is actually <u>absolutely continous</u>, so in particular it is differentiable a.e., and one has $(Vf)'(x) = f(x)$ for a.e. $x \in [0,1]$. This is one part of the general fundamental theorem of calculus.

(3) <u>V is compact</u>

There are many possibilities to show this, e.g.

- use ex. 45 (ii) => (i).
- use that by (1) V can be decomposed as

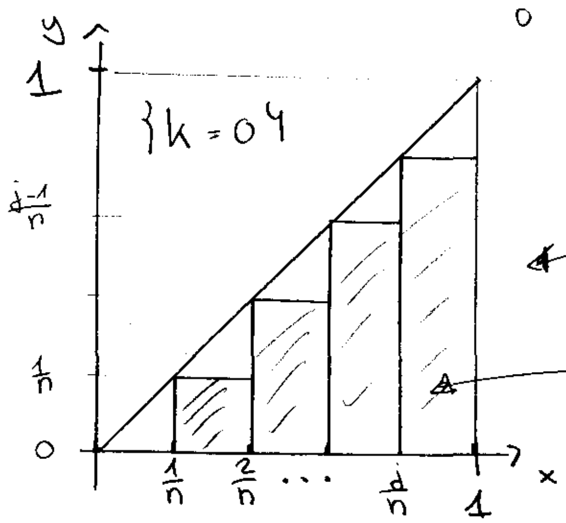
$$L^2([0,1]) \xhookrightarrow[\text{inclusion}]{\text{canonical}} L^1([0,1]) \xrightarrow{\tilde{V}} C([0,1]) \xhookrightarrow[\text{inclusion}]{\text{canonical}} L^2([0,1]),$$

  and $\tilde{V}: L^1([0,1]) \to C([0,1])$ is cpt. by the Arzela-Ascoli Thm. (convince yourself that the assumptions for A.-A. are easily verified)

- We approximate V by finite rank operators by approximating its kernel by step functions on rectangles in $L^2([0,1]^2)$.

We show here only the third possibility: (the other are left as excise for the interested student)

<u>idea</u>: $Vf(x) = \int_0^1 \underbrace{\mathbb{1}_{[0,x]}(y)}_{=: k(x,y)} f(y)\,dy$



{k = 1}

{k_n = 1}, Approximating the triangle by rectangels

Formally, for all $n \in \mathbb{N}$ we define

$$k_n(x,y) := \sum_{j=1}^{n} \mathbb{1}_{(\frac{j-1}{n}, \frac{j}{n}] \times [0, \frac{j-1}{n}]}(x,y)$$

$$= \sum_{j=1}^{n} \mathbb{1}_{(\frac{j-1}{n}, \frac{j}{n}]}(x) \cdot \mathbb{1}_{[0, \frac{j-1}{n}]}(y) \quad \text{for all } x,y \in [0,1]$$

Then: (i) $k_n \longrightarrow k$ pointwise a.e. for $n \to \infty$.

proof: Let $N := \{(x,x) \mid x \in [0,1]\} \subseteq [0,1]^2$, then N is a nullset. Let $(x,y) \in [0,1]^2 \setminus N$.

Case 1: $y > x$, then $(x,y) \notin [\frac{j-1}{n}, \frac{j}{n}] \times [0, \frac{j-1}{n}]$, hence $k_n(x,y) = 0 = k(x,y)$ for all $n \in \mathbb{N}$.

Case 2: $y < x$, then choose $n_0 \in \mathbb{N}$ with $\frac{1}{n_0} < x - y$. Let $n \geq n_0$.
Choose $j \in \{1, \dots, n\}$ with $x \in (\frac{j-1}{n}, \frac{j}{n}]$, then

$$y < x - \frac{1}{n} \leq \frac{j}{n} - \frac{1}{n} = \frac{j-1}{n},$$

hence $(x,y) \in (\frac{j-1}{n}, \frac{j}{n}] \times [0, \frac{j-1}{n}]$, so

$$k_n(x,y) = 1 = k(x,y) \quad \text{for all } n \geq n_0.$$

(ii) $|k_n| \leq \mathbb{1}_{[0,1]^2}$ for all $n \in \mathbb{N}$, and $\mathbb{1}_{[0,1]^2} \in L^2([0,1]^2)$.
By Lebesgue's thm., we have $k_n \xrightarrow{n \to \infty} k$ in $L^2([0,1]^2)$.

Define $T_{k_n}: L^2([0,1]) \to L^2([0,1])$, $f \mapsto \int_0^1 k_n(\cdot, y) f(y)\, dy$,
Then

$$T_{k_n} f(x) = \int_0^1 \sum_{j=1}^{n} \mathbb{1}_{(\frac{j-1}{n}, \frac{j}{n}]}(x)\, \mathbb{1}_{[0, \frac{j-1}{n}]}(y) f(y)\, dy$$

$$= \sum_{j=1}^{n} \left(\int_0^{\frac{j-1}{n}} f(y)\, dy \right) \cdot \mathbb{1}_{(\frac{j-1}{n}, \frac{j}{n}]}(x),$$

hence $T_{k_n}(L^2([0,1])) \subseteq \operatorname{span}\{\mathbb{1}_{(\frac{j-1}{n}, \frac{j}{n}]} \mid j \in \{1, \dots, n\}\}$.
So each T_{k_n} is a finite rank operator, hence cpt., and

$$\| T_{k_n} - V \|_{B(H)} = \| T_{k_n - k} \| \underset{\substack{\uparrow \\ \text{older ex. or lecture}}}{\leq} \| k_n - k \|_{L^2} \xrightarrow{n \to \infty} 0,$$

So V is cpt. as the limit of finite rank operators.

a) <u>Claim</u>: R is compact and selfadjoint, and it has the representation

$$Rf(x) = \int_0^x y f(y)\,dy + x \int_x^1 f(y)\,dy.$$

<u>Proof</u>. Since V is cpt, $R = VV^*$ is cpt. as well.
Moreover, $R^* = (VV^*)^* = (V^*)^* V^* = VV^* = R$.
We already observed that V is a kernel operator with kernel $k(x,y) = \mathbb{1}_{[0,x]}(y)$ for all $x, y \in [0,1]$.
So ex. 44 b) says, that V^* is also a kernel operator with kernel $k^*(x,y) = k(y,x)$ for all $x, y \in [0,1]$,
and ex. 44 a) says, that $R = VV^*$ is again a kernel operator with kernel

$$h(x,y) = \int_0^1 k(x,z)\,k^*(z,y)\,dz = \int_0^1 \mathbb{1}_{[0,x]}(z)\,\mathbb{1}_{[0,y]}(z)\,dz$$
$$= \int_0^{\min\{x,y\}} dz = \min\{x,y\} \quad \text{for all } x, y \in [0,1].$$

So

$$Rf(x) = \int_0^1 h(x,y) f(y)\,dy = \int_0^1 \min\{x,y\} f(y)\,dy$$
$$= \int_0^x y f(y)\,dy + x \int_x^1 f(y)\,dy.$$

b) <u>Claim</u>: The eigenvalues of R are given by $\alpha_n = \frac{4}{\pi^2} \frac{1}{(2n-1)^2}$, $n \in \mathbb{N}$, and the corresponding eigenspaces are one-dimensional and spanned by

$$\varepsilon_n = \sqrt{2} \cdot \sin\left(\frac{2n-1}{2}\pi\,\cdot\right)\Big|_{[0,1]}.$$

<u>Proof / Construction</u>: We make the

<u>Ansatz</u>: $Rf = \alpha f$ with $\alpha \in \mathbb{R}$, $f \neq 0$.

Then:

$$(*)\begin{cases}\alpha \underbrace{\langle f, f\rangle}_{>0} = \langle \alpha f, f\rangle = \langle Rf, f\rangle = \langle VV^* f, f\rangle \\ = \langle V^* f, V^* f\rangle = \|V^* f\|^2 \geq 0,\end{cases}$$

hence $\underline{\alpha \geq 0}$.

Case 1 $\alpha = 0$. Then $(*)$ shows also $\|V^* f\|^2 = 0$, so $V^* f = 0$, i.e.

$$0 = V^* f = \int_\cdot^1 f(y)\,dy = \int_0^1 f(y)\,dy - Vf.$$

By (2), Vf is weakly db with weak derivative f, so

$$0 = (V^* f)' = 0 - (Vf)' = -f.$$

Case 2 $\alpha \neq 0$. Then

$$f = \frac{1}{\alpha} Rf = \frac{1}{\alpha}\Big(V\big(\underbrace{id_{[0,1]}\cdot f}_{\in L^2([0,1])}\big) + id_{[0,1]}\cdot\Big(\int_0^{\hat{}} f(y)\,dy - Vf\Big)\Big),$$

so by (1), f is continous (because the Right-hand-side is continous), and by inductive use of the fundamental theorem of calculus (for continous functions), we see that f is even C^∞, in particular $f \in C^2([0,1])$.

From the integral equation

$$\alpha f(x) = \int_0^x y f(y)\,dy + x\int_x^1 f(y)\,dy$$

we derive:

$$\alpha f'(x) = \cancel{x f(x)} + \int_x^1 f(y)\,dy - \cancel{x f(x)},$$

$$\begin{cases}\alpha f''(x) = -f(x), \\ f(0) = 0, \\ f'(1) = 0.\end{cases} \qquad (BVP)$$

Hence, f is a solution of (BWP).

By Analysis III, ODE:

$$f'' = -\frac{1}{\alpha} f$$

$$\rightsquigarrow \quad f(x) = A \sin\left(\frac{1}{\sqrt{\alpha}} x\right) + B \cos\left(\frac{1}{\sqrt{\alpha}} x\right);$$

bound. values: $0 = f(0) = A\sin(0) + B\cos(0) = B$

$$\Rightarrow B = 0,$$

hence $f(x) = A \sin\left(\frac{1}{\sqrt{\alpha}} x\right), \quad A \neq 0.$

second Bound cond. $0 = f'(1) = \frac{A}{\sqrt{\alpha}} \cos\left(\frac{1}{\sqrt{\alpha}}\right)$

$$\Rightarrow \exists n \in \mathbb{N}: \frac{1}{\sqrt{\alpha}} = \frac{2n-1}{2}\pi$$

$$\Rightarrow \quad \text{- " -} \quad \alpha = \alpha_n$$

and in this case $f = \frac{A}{\sqrt{2}} \varepsilon_n$.

So we have shown: $\sigma_p(R) \subseteq \{\alpha_n \mid n \in \mathbb{N}\}$ and $\dim \ker(\alpha_n I - R) \leq 1$ for all $n \in \mathbb{N}$.

On the other hand, for all $n \in \mathbb{N}$ we have

$$\frac{1}{\sqrt{2}} R \varepsilon_n(x) = \int_0^x y \sin(\alpha_n^{-\frac{1}{2}} y)\,dy + x \int_x^1 \sin(\alpha_n^{-\frac{1}{2}} y)\,dy$$

$$= \sqrt{\alpha_n}\, y \left(-\cos\left(\frac{y}{\sqrt{\alpha_n}}\right)\right)\Big|_0^x + \sqrt{\alpha_n} \int_0^x \cos\left(\frac{y}{\sqrt{\alpha_n}}\right) dy - x\sqrt{\alpha_n} \cos\left(\frac{y}{\sqrt{\alpha_n}}\right)\Big|_x^1$$

$$= -\sqrt{\alpha_n}\, x \cos\left(\frac{x}{\sqrt{\alpha_n}}\right) + \alpha_n \sin\left(\frac{y}{\sqrt{\alpha_n}}\right)\Big|_0^x - \underbrace{x\sqrt{\alpha_n} \cos\left(\frac{1}{\sqrt{\alpha_n}}\right)}_{=0} + x\sqrt{\alpha_n}\cos\left(\frac{x}{\sqrt{\alpha_n}}\right)$$

$$= \alpha_n \sin\left(\frac{x}{\sqrt{\alpha_n}}\right) = \frac{1}{\sqrt{2}} \alpha_n \varepsilon_n(x),$$

hence $R\varepsilon_n = \alpha_n \varepsilon_n$. //

c) Claim: $(\varepsilon_n)_{n\in\mathbb{N}}$ is an ONB of H.

proof: By the spectral thm. for cpt s.a operators,

$$H = \underbrace{\ker K}_{=\{0\}} \oplus \overline{\operatorname{span}\{\varepsilon_n : n \in \mathbb{N}\}} = \overline{\operatorname{span}\{\varepsilon_n : n \in \mathbb{N}\}},$$

since $0 \notin \sigma_p(K)$

and since K is s.a., the ε_n are orthogonal.

Finally, $\|\varepsilon_n\|_H^2 = 2\int_0^1 \sin^2\left(\frac{2n-1}{2}\pi x\right) dx = \dots = 1$. //