**Functional Analysis**

**Exercise Sheet 12**

*Remark.* The exercises marked with a * can be handed in for correction in the “Funktionalanalysis” box in the atrium of building 20.30 at the latest at 14:00 on the day of the exercise class next week.

**Exercise 1: Examples of adjoint operators**

(a) Let $A \in \mathbb{R}^{n \times n}$ be a matrix viewed as a map $A : \mathbb{R}^n \to \mathbb{R}^n$. Show that $A' = A^t$, i.e., the adjoint operator of $A$ is given by the transpose matrix of $A$.

(b) Let $p \in (1, \infty)$ and define $p' \in (1, \infty)$ through the relation $\frac{1}{p} + \frac{1}{p'} = 1$. Suppose for each $j, k \in \mathbb{N}$ we are given an $a_{j,k} \in \mathbb{K}$ with

$$\left( \sum_{j=1}^{\infty} \left( \sum_{k=1}^{\infty} |a_{j,k}|^{p'} \right)^{\frac{p}{p'}} \right)^{\frac{1}{p}} < \infty,$$

and consider the operator $T : \ell^p \to \ell^p$, $Tx := (\sum_{k=1}^{\infty} a_{j,k} x_k)_{j \in \mathbb{N}}$ from Exercise 3 of Exercise sheet 6. Determine $T'$.

(c) Let $T : L^2([0, 1]) \to L^2([0, 1])$, $Tf(x) := \int_0^x f(t) \, dt$. Show that $T \in L(L^2([0, 1]), L^2([0, 1]))$ and determine $T'$.

(d) Define $T : L^1([0, 1]) \to c_0$ by $Tf := \left( \int_0^1 f(t) t^j \, dt \right)_{j=0}^{\infty}$. Show that $T \in L(L^1([0, 1]), c_0)$ and determine $T'$.

* Exercise 2: Properties of adjoint operators

Let $X, Y$ be Banach spaces and let $T \in L(X, Y)$.

(a) Show that $R(T)$ is dense in $Y$ if and only if $T'$ is injective.

(b) Show that if $T$ is injective and $R(T)$ is closed in $Y$, then $T'$ is surjective.

(c) Show that $T$ is invertible if and only if $T'$ is invertible.

* Exercise 3: Reflexivity

(a) Prove that any Hilbert space is reflexive.
(b) Show that the following spaces are not reflexive:

(i) $\ell^1, L^1([0,1])$;

(ii) $\ell^\infty, L^\infty([0,1])$;

(iii) $C([0,1])$;

(Hint: show that $\|ev_x - ev_y\| = 2$ in $(C([0,1]))'$ for any distinct $x, y \in [0,1]$, where $ev_x(\phi) = \phi(x)$. Alternatively, find an isometry $c_0 \rightarrow C([0,1])$.)

(iv) $L(p)$ for $p \in [1, \infty]$.

(Hint: use Exercise 2 of exercise sheet 4).

Exercise 4: Reflexivity in spaces of differentiable functions

(a) Let $p \in (1, \infty)$. Show that there is an isometry $W^{1,p}([0,1]) \rightarrow L^p([0,1]) \times L^p([0,1])$. Conclude that $W^{1,p}([0,1])$ is reflexive.

(b) Show that $C^1([0,1])$ is isomorphic to $C([0,1]) \times \mathbb{K}$. Conclude that $C^1([0,1])$ is not reflexive.