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Functional Analysis

Solutions to exercise sheet 2

Exercise 1: Relative metrics
Let $(X, d)$ be a metric space and let $M \subseteq X$ be equipped with the relative metric $d_M$, i.e., the restriction of $d$ to $M \times M$.

(a) Show that $A \subseteq M$ is open in $M$ if and only if there exists a set $A' \subseteq X$ that is open in $X$ satisfying $A = A' \cap M$. Prove an analogous result for closed sets.

(b) For a subset $A \subseteq M$ we denote the closure of $A$ with respect to $d_M$ as $\overline{A}^M$ and the closure with respect to $d$ as $\overline{A}$. Show that $\overline{A}^M = \overline{A} \cap M$.

(c) (Tricky) Suppose that $X$ is separable. Prove that $M$ is also separable.

Solution: We begin with a general observation about balls with respect to $d_M$. We denote the ball centered at $x \in M$ with radius $r > 0$ with respect to $d_M$ by $B_M(x, r)$, while we denote this ball with respect to $d$ by $B(x, r)$. Then

$$ B_M(x, r) = \{y \in M : d_M(x, y) < r\} $$

$$ = \{y \in X : y \in M \text{ and } d(x, y) < r\} $$

$$ = M \cap B(x, r). $$

(1)

(a) “$\Leftarrow$”

Suppose $A = A' \cap M$ for some set $A'$ that is open in $(X, d)$. We wish to show that $A$ is open in $(M, d_M)$. To this end, let $x \in A$. Our goal is to find an $r > 0$ such that $B_M(x, r) \subseteq A$. Since also $x \in A'$, so we can find an $r > 0$ such that $B(x, r) \subseteq A'$. But then by (1) we have

$$ B_M(x, r) = M \cap B(x, r) \subseteq M \cap A' = A. $$

Thus, $A$ is open with respect to $d_M$, as desired.

“$\Rightarrow$”

Suppose $A \subseteq M$ is open with respect to $d_M$. Then for each $x \in A$ we can find an $r_x > 0$ such that $B_M(x, r_x) \subseteq A$. We claim that

$$ A = \bigcup_{x \in A} B_M(x, r_x). $$

(2)

Indeed, the inclusion $A \subseteq \bigcup_{x \in A} B_M(x, r_x)$ follows from the fact that each $x \in A$ lies in the ball $B(x, r_x)$. For the converse inclusion, let $y \in \bigcup_{x \in A} B_M(x, r_x)$. Then there is some $x \in A$ such that $y \in B_M(x, r_x)$. Since $B_M(x, r_x) \subseteq A$, we also have $y \in A$. This proves the claim.

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We set \( A' := \bigcup_{x \in A} B(x, r_x) \). Since the balls \( B(x, r_x) \) are open in \((X, d)\), the set \( A' \) is also open in \((X, d)\) as it is a union of open sets. Now note that by (1) and (2) we have

\[
A' \cap M = \bigcup_{x \in A} B(x, r_x) \cap M = \bigcup_{x \in A} B_M(x, r_x) = A.
\]

The result follows.

For the result about closed sets, we wish to show that \( B \subseteq M \) is closed in \( M \) if and only if there is a set \( B' \subseteq X \) that is closed in \( X \) such that \( B' \cap M = B \).

To prove this result, note that \( B \subseteq M \) is closed with respect to \( d_M \) if and only if \( M \setminus B \) is open with respect to \( d_M \). Thus, by what we have shown about open sets, this is equivalent to the existence of a set \( A' \subseteq X \) that is open in \( X \) such that \( M \setminus B = A' \cap M \). But then for \( B' := X \setminus A' \), which is closed in \( X \), we have

\[
B' \cap M = X \setminus A' \cap M = M \setminus A' = M \setminus (A' \cap M) = M \setminus (M \setminus B) = B.
\]

Thus, the desired equivalence follows.

(b) By part (a), we have the following equality of collections of sets:

\[
\{ F : F \text{ is a closed subset of } (M, d_M) \} = \{ F' \cap M : F' \text{ is a closed subset of } (X, d) \}
\]

Thus\(^1\), we have

\[
\overline{A} \cap M = \bigcap_{F' \supseteq A} F' \cap M = \bigcap_{F' \supseteq A} F' = \bigcap_{F \supseteq A} F = A^M.
\]

(c) Let \( D \) be a countable dense subset of \( X \). Denoting the set of positive rational numbers by \( \mathbb{Q}_+ \), for each rational \( r \in \mathbb{Q}_+ \) and each \( x \in D \) we check if \( B(x, r) \cap M \) is empty or not. If it is not empty, then we choose\(^2\) a point from it. We denote the collection of these points by \( D_M \). Note that \( D_M \) is countable, since it has at most as many elements as \( \mathbb{Q}_+ \times D \).

We claim that the countable set \( D_M \) is dense in \( M \). Since \( \overline{D_M}^M \subseteq M \), it remains to show that \( M \subseteq \overline{D_M}^M \). Let \( y \in M \) and let \( \varepsilon > 0 \). Pick \( r \in \mathbb{Q}_+ \) such that \( 0 < r < \frac{\varepsilon}{2} \). Since \( D \) is dense in \( X \), the ball \( B(y; r) \) intersects \( D \) and thus, we may pick a point \( x \in D \cap B(y; r) \). Then the set \( B(x, r) \cap M \) is not empty as it contains the point \( y \). Thus, by construction, there is a point \( z \in D_M \) that lies in \( B(x, r) \cap M \). It remains to note that \( z \in B_M(y, \varepsilon) \).

Indeed, by the triangle inequality we have

\[
d(z, y) \leq d(z, x) + d(x, y) < 2r < \varepsilon.
\]

Thus, we have shown that \( B_M(y, \varepsilon) \cap D_M \) is non-empty. This proves that \( y \in \overline{D_M}^M \) and thus \( \overline{D_M}^M = M \).

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\(^1\)Here we are using the fact that the closure of a set \( C \) in a metric space \((Y, \rho)\) is the intersection of all closed sets in \( Y \) containing \( C \), c.f., Exercise 3(d) of Exercise sheet 0.

\(^2\)We are using the axiom of choice here.
**Exercise 2: Separability in normed spaces**

Let $(X, \| \cdot \|)$ be a normed space. Show that the following are equivalent:

(i) $X$ is separable;

(ii) The unit ball $B_X := \{ x \in X : \| x \| < 1 \}$ is separable;

(iii) The unit sphere $S_X := \{ x \in X : \| x \| = 1 \}$ is separable.

**Solution:** Our strategy will be to show that (i)$\iff$(ii) and (i)$\iff$(iii). Note that in both cases “$\Rightarrow$” follows from Exercise 1(c). Thus it remains to prove the “$\Leftarrow$” cases.

“(i)$\Leftarrow$(ii)”

Let $D$ be a countable dense subset in $B_X$. By a result in the lecture, it suffices to show that $\text{span} \ D$ is dense in $X$ to conclude that $X$ is separable. Let $x \in X$. As $0 \in \text{span} \ D$, we may assume that $x \neq 0$. Then set $y := \frac{x}{2\|x\|}$. Then $\|y\| = \frac{1}{2} < 1$, so $y \in B_X$. Since $D$ is dense in $B_X$, we can find a sequence $(y_n)_{n \in \mathbb{N}}$ in $B_X$ such that $y_n \to y$ as $n \to \infty$. Now set $x_n := 2\|x\|y_n$. Then $(x_n)_{n \in \mathbb{N}}$ is a sequence in $\text{span} \ D$. Moreover, we have

$$\|x_n - x\| = 2\|x\|\|y_n - y\| \to 0, \quad \text{as } n \to \infty.$$ 

Hence, we have shown that any point in $X$ is the limit of a sequence in $\text{span} \ D$. Thus, $\overline{\text{span} \ D} = X$, as desired.

“(i)$\Leftarrow$(iii)”

The proof is completely analogous to the proof of “(i)$\Leftarrow$(ii)”, except that we now take $y := \frac{x}{\|x\|}$ and $x_n = \|x\|y_n$.

**Exercise 3: Separability and density under continuous maps**

Let $(X, d)$ and $(Y, \rho)$ be metric spaces and let $f : X \to Y$ be continuous.

(a) Suppose that $A \subseteq X$ is dense in $X$. Show that $f(A)$ is dense in $f(X)$.

(b) Show that if $X$ is separable, then so is $f(X)$.

**Solution:**

(a) To prove this, it is probably easiest to use sequences. However, we give a proof that works in a more general context here.

For this exercise one can use the implication\(^3\) that if $f$ is continuous, then $f(\overline{B}^X) \subseteq \overline{f(B)}^Y$ for any set $B \subseteq X$. Here, for clarity, we are writing the space in which we are taking the closure as a superscript.

To prove this implication, note that $f(\overline{B}^X) \subseteq \overline{f(B)^Y}$ is equivalent to the inclusion $\overline{B}^X \subseteq f^{-1}(\overline{f(B)}^Y)$. But to prove the latter, one can note that since $f$ is continuous and $\overline{f(B)}^Y$

\(^3\)Actually, this is an equivalence!
is closed in \( Y \), the set \( f^{-1}(f(B)^Y) \) is closed in \( X \). Since \( B \) is a subset of \( f^{-1}(f(B)^Y) \), also \( \overline{B}^X \subseteq f^{-1}(f(B)^Y) \), as desired.

Now we will prove that if \( A \) is dense in \( X \), then \( f(A) \) is dense in \( f(X) \). Indeed, since \( \overline{A}^X = X \), our implication implies that \( f(X) = f(\overline{A}^X) \subseteq f(A)^Y \). But then, by Exercise 1(b), we have
\[
\overline{f(A)}^{f(X)} = \overline{f(A)}^Y \cap f(X) \supseteq f(X) \cap f(X) = f(X).
\]
Thus, since also \( \overline{f(A)}^{f(X)} \subseteq f(X) \), we have \( \overline{f(A)}^{f(X)} = f(X) \), as asserted.

(b) Let \( D \) be a countable dense subset in \( X \). Then \( f(D) \) is a countable subset of \( f(X) \). Moreover, by part (a), it is dense. We conclude that \( f(X) \) is separable.

**Exercise 4: Failure of completeness of \( \mathbb{R} \) with an equivalent metric**
Consider the real line \( \mathbb{R} \) equipped with the metric \( d(x, y) := |\arctan(x) - \arctan(y)| \).

(a) Let \( (x_n)_{n \in \mathbb{N}} \subseteq \mathbb{R} \) be a sequence and \( x \in \mathbb{R} \). Show that \( \lim_{n \to \infty} d(x_n, x) = 0 \) if and only if \( \lim_{n \to \infty} |x_n - x| = 0 \), i.e., show that \( d \) is equivalent to the usual metric on \( \mathbb{R} \).

(b) Show that \( (\mathbb{R}, d) \) is not complete.

**Solution:**

(a) We first prove the direct implication. Suppose \( d(x_n, x) \to 0 \) as \( n \to \infty \). Then this is equivalent to saying that the sequence \( (\arctan(x_n))_{n \in \mathbb{N}} \) converges to \( \arctan(x) \) with respect to the usual metric on \( \mathbb{R} \). Since \( \tan \) is continuous, this means that \( \lim_{n \to \infty} x_n = \lim_{n \to \infty} \tan(\arctan(x_n)) = \tan(\arctan(x)) = x \) with respect to the usual metric on \( \mathbb{R} \), i.e., \( \lim_{n \to \infty} |x_n - x| = 0 \).

The converse implication is proven analogously, this time using the fact that \( \arctan \) is continuous.

(b) Consider the sequence \( (n)_{n \in \mathbb{N}} \). Then, since
\[
\lim_{m, n \to \infty} |\arctan(n) - \arctan(m)| \leq \lim_{m, n \to \infty} |\arctan(n) - \frac{\pi}{2}| + |\frac{\pi}{2} - \arctan(m)| = 0,
\]
this sequence is a Cauchy sequence with respect to \( d \). However, this sequence is not convergent with respect to \( d \). Indeed, if it were convergent with respect to \( d \), then by part (a), the sequence \( (n)_{n \in \mathbb{N}} \) would converge in \( \mathbb{R} \) with respect to its usual metric, which is absurd.

Thus, we conclude that \( (\mathbb{R}, d) \) is not complete.

**Exercise 5: Point evaluations on \( C([0, 1]) \)**
Consider the space \( C([0, 1]) \) equipped with the supremum norm \( \| \cdot \|_\infty \).

(a) Let \( x \in [0, 1] \) and define the evaluation map \( \text{ev}_x : C([0, 1]) \to \mathbb{R} \) by \( \text{ev}_x(f) := f(x) \). Prove that \( \text{ev}_x \) is continuous.

(b) Using the evaluation maps, prove that the following sets are closed in \( C([0, 1]) \):

(i) \( A = \{ f \in C([0, 1]) : f(x) \in F \} \), where \( x \in [0, 1] \) and \( F \subseteq \mathbb{R} \) is a closed set.
(ii) \( B = \{ f \in C([0,1]) : f(x) \geq 0 \text{ for all } x \in [0,1] \} \).

**Solution:**

(a) Note that
\[
|ev_x(f) - ev_x(g)| = |f(x) - g(x)| \leq \|f - g\|_\infty.
\]

Thus, \( ev_x \) is even Lipschitz continuous.

To be more precise, we can let \( \varepsilon > 0 \) and choose \( \delta := \varepsilon \). Then if \( \|f - g\|_\infty < \delta \), it follows from the inequality (3) that \( |ev_x(f) - ev_x(g)| < \varepsilon \), proving that \( ev_x \) is continuous.

Alternatively, one could use the fact that if a sequence \( (f_n)_{n \in \mathbb{N}} \) in \( C([0,1]) \) converges to a function \( f \in C([0,1]) \) with respect to \( \| \cdot \|_\infty \), then (3) implies that \( ev_x(f_n) \to ev_x(f) \) in \( \mathbb{R} \), again proving that \( ev_x \) is continuous.

(b) For (i), note that since \( ev_x \) is continuous and \( F \) is closed, the set
\[
ev_x^{-1}(F) = \{ f \in C([0,1]) : ev_x(f) \in F \} = \{ f \in C([0,1]) : f(x) \in F \} = A
\]
is again closed, proving the result.

For (ii) we first note that since the set \([0,\infty)\) is closed in \( \mathbb{R} \), the set
\[
ev_x^{-1}([0,\infty)) = \{ f \in C([0,1]) : ev_x(f) \in [0,\infty) \} = \{ f \in C([0,1]) : f(x) \geq 0 \}
\]
is also closed. Thus, since intersections of closed sets are again closed, the set
\[
\bigcap_{x \in [0,1]} ev_x^{-1}([0,\infty)) = \{ f \in C([0,1]) : f(x) \geq 0 \text{ for all } x \in [0,1] \} = B
\]
is closed. The assertion follows.