Exercise 1: Totally boundedness

(a) Suppose that \((X, d)\) is a totally bounded metric space. Show that \(X\) is separable.

(b) Let \((x_n)_{n \in \mathbb{N}}\) be a Cauchy sequence in a metric space \((X, d)\). Show that the set \(\{x_n : n \in \mathbb{N}\}\) is totally bounded.

(c) Show that a subset of \(\mathbb{R}^n\) is totally bounded if and only if it is bounded.

(d) Equip \(\mathbb{R}\) with the metric \(d(x, y) := \min\{1, |x - y|\}\). Show that \((\mathbb{R}, d)\) is bounded, but not totally bounded.

Solution:

(a) For each \(n \in \mathbb{N}\) we can find a finite collection of points \(F_n \subseteq X\) such that

\[ X \subseteq \bigcup_{x \in F_n} B(x, \frac{1}{n}). \tag{1} \]

Now set \(D := \bigcup_{n \in \mathbb{N}} F_n\). As a countable union of finite sets, this set is again countable. To prove that \(X\) is separable, it remains to show that \(D\) is dense in \(X\).

Let \(y \in X\) and \(\varepsilon > 0\). To conclude that \(\overline{D} = X\), we need to show that \(B(y, \varepsilon) \cap D \neq \emptyset\). Pick \(n \in \mathbb{N}\) large enough such that \(\frac{1}{n} < \varepsilon\). Then by (1), we can find an \(x \in F_n\) such that \(y \in B(x, \frac{1}{n})\). Then \(d(x, y) < \frac{1}{n} < \varepsilon\) so that \(x \in B(y, \varepsilon)\). Since also \(x \in D\), we conclude that \(B(y, \varepsilon) \cap D \neq \emptyset\), as desired. The assertion follows.

(b) Let \(\varepsilon > 0\) and let \(N \in \mathbb{N}\) such that \(d(x_n, x_m) < \varepsilon\) whenever \(n, m \geq N\). In particular, this implies that \(x_n \in B(x_N, \varepsilon)\) for all \(n \geq N\). Thus, since \(x_n \in B(x_n, \varepsilon)\) for \(n \in \{1, \ldots, N\}\), we have

\[ \{x_n : n \in \mathbb{N}\} \subseteq \bigcup_{n=1}^{N} B(x_n, \varepsilon). \]

Thus, \(\{x_n : n \in \mathbb{N}\}\) is totally bounded, as asserted.

(c) The implication that a totally bounded set is bounded is true in general. The difficult direction to prove is that bounded sets in \(\mathbb{R}^n\) are totally bounded. One might be tempted to use the fact that the closure of a bounded set in \(\mathbb{R}^n\) is compact and thus, since compact sets are in particular totally bounded, the result follows. This argument is however circular, since the proof of the fact that closed and bounded sets in \(\mathbb{R}^n\) are compact makes use of the fact that bounded sets in \(\mathbb{R}^n\) are totally bounded. Thus, one should prove the directly.
We first prove this for $\mathbb{R}$. Suppose $A \subseteq \mathbb{R}$ is a bounded. Then there is some $r > 0$ such that $A \subseteq (-r, r)$. It now suffices to show that $(-r, r)$ is totally bounded to conclude that $A$ is, since subsets of totally bounded sets are totally bounded.

Let $\varepsilon > 0$. Then consider the set of points $F := \{n \varepsilon : n \in \mathbb{Z}\} \cap (-r, r)$ and note that $F$ is finite (check this). Moreover, we have $(-r, r) \subseteq \bigcup_{t \in F} B(t, \varepsilon)$. Indeed, if $x \in (-r, r)$, then there is a unique $n \in \mathbb{Z}$ such that $n \varepsilon \leq x < (n + 1)\varepsilon$. In particular, this implies that $|x - n\varepsilon| < \varepsilon$ and that $|x - (n + 1)\varepsilon| < \varepsilon$. Since either $n\varepsilon$ (when $x \geq 0$) or $(n + 1)\varepsilon$ (when $x < 0$) lies in $(-r, r)$, we conclude that $x \in \bigcup_{t \in F} B(t, \varepsilon)$. The assertion follows.

Now that we have established the result for $\mathbb{R}$, the general result follows from the following result:

**Lemma 1.** Let $(X, d)$ and $(Y, \rho)$ be metric spaces and let $A \subseteq X$ and $B \subseteq Y$ be totally bounded. Then $A \times B$ is also totally bounded with respect to any of the equivalent metrics

\[
d_p((x, y), (x', y')) = \left(d(x, x')^p + \rho(y, y')^p\right)^{\frac{1}{p}} \quad \text{for } p \in [1, \infty) \\
d_\infty((x, y), (x', y')) = \max\{d(x, x'), \rho(y, y')\}
\]

on $X \times Y$.

Indeed, using this lemma with the $p = 2$ product metric allows us to use induction on the dimension $n$ to conclude that any set of the form $(-r, r)^n \subseteq \mathbb{R}^n$ for $r > 0$ is totally bounded. Since any bounded set $A \subseteq \mathbb{R}^n$ lies in such a set, any bounded set is totally bounded.

**Proof of Lemma 1.** We will prove the result for the product metric $d_\infty$. The other cases follow from this case and the fact that for any $p \in [1, \infty)$ we have

\[
d_p((x, y), (x', y')) \leq 2^\frac{1}{p}d_\infty((x, y), (x', y')).
\]

Indeed, from this estimate it follows that if the balls $B_{d_\infty}((x_1, y_1), 2^{-\frac{1}{p}}\varepsilon), \ldots, B_{d_\infty}((x_N, y_N), 2^{-\frac{1}{p}}\varepsilon)$ covers $A \times B$, then so do the balls $B_{dp}((x_1, y_1), \varepsilon), \ldots, B_{d_\infty}((x_N, y_N), \varepsilon)$, as desired.

The metric $d_\infty$ works nicely with respect to products, since in this case

\[
B_{d_\infty}((x, y), r) = B_d(x, r) \times B_\rho(y, r).
\]

Indeed, we have $(x', y') \in B_{d_\infty}((x, y), r)$ if and only if $\max\{d(x, x'), \rho(y, y')\} < r$ if and only if $d(x, x') < r$ and $\rho(y, y') < r$ if and only if $(x', y') \in B_d(x, r) \times B_\rho(y, r)$, proving the equality.

Now let $\varepsilon > 0$. Since both $A$ and $B$ are totally bounded, we can find finite sets $F \subseteq A$ and $G \subseteq B$ such that $A \subseteq \bigcup_{x \in F} B_d(x, \varepsilon)$ and $B \subseteq \bigcup_{y \in G} B_\rho(y, \varepsilon)$. But then

\[
A \times B \subseteq \bigcup_{x \in F, y \in G} B_d(x, \varepsilon) \times B_\rho(y, \varepsilon) = \bigcup_{(x, y) \in F \times G} B_{d_\infty}((x, y), \varepsilon).
\]

Since $F \times G$ is finite, this proves that $A \times B$ is totally bounded, proving the lemma.

**Exercise 2: Compactness**

(a) Let $(X, d)$ be a compact metric space and let $A \subseteq X$ be closed. Show that $A$ is compact.
(b) Let \((X, d)\) be a metric space and let \(B \subseteq X\) be relatively compact. Show that any subset \(A \subseteq B\) is also relatively compact.

(c) Let \((X, d), (Y, \rho)\) be metric spaces and suppose that \(X\) is compact. Show that if \(f : X \to Y\) is a continuous bijective map, then its inverse \(f^{-1} : Y \to X\) is also continuous.

(d) Let \((X, d), (Y, \rho)\) be compact metric spaces. Show that \(X \times Y\) is also compact.

**Solution:** For all of these result it is possible to both use a proof using open sets and a proof using sequences.

(a) The proof using sequences here is completely straightforward. We present the proof using open covers here.

Suppose \((U_i)_{i \in I}\) is an open cover of \(A\). Since \(A\) is closed, the set \(X \setminus A\) is open in \(X\). This means that the collection \((U_i)_{i \in I} \cup \{X \setminus A\}\) is an open cover of \(X\). Since \(X\) is compact, we can find a finite \(F \subseteq I\) such that \((U_i)_{i \in F} \cup \{X \setminus A\}\) covers \(X\). But then \((U_i)_{i \in F}\) is a finite subcover of \(A\), proving that \(A\) is compact.

(b) This follows from part (a). Indeed, if \(B\) is relatively compact this means per definition that \(\overline{B}\) is compact. Since \(\overline{A} \subseteq \overline{B}\), the set \(\overline{A}\) is a closed subset of the compact set \(\overline{B}\). By part (a), this implies that \(\overline{A}\) is compact. Hence, \(A\) is a relatively compact, as asserted.

(c) To show that \(f^{-1}\) is continuous, we will show that for each closed set \(F \subseteq X\) we have that \((f^{-1})^{-1}(F) = f(F)\) is closed in \(Y\).

Let \(F \subseteq X\) be closed. Then, since \(X\) is compact, it follows from part (a) that \(F\) is compact. Since the continuous image of a compact set is again compact, the set \(f(F)\) is compact in \(Y\). Finally, since compact sets are closed, we conclude that \(f(F)\) is closed, as desired. This proves the continuity of \(f^{-1}\).

Alternatively, one can prove this with sequences. Indeed, suppose \((y_n)_{n \in \mathbb{N}}\) is a sequence in \(Y\) with limit \(y \in Y\). We conclude that \(f^{-1}\) is continuous, we need to show that \(f^{-1}(y_n) \to f^{-1}(y)\) in \(X\). Writing \(x_n := f^{-1}(y_n), x := f^{-1}(y)\), this means it remains to show that \(x_n \to x\) in \(X\). For this we will use Exercise 1(b) of Exercise sheet 0. Thus, we need to show that any subsequence \((x_{n_j})_{j \in \mathbb{N}}\) has a further subsequence \((x_{n_{jk}})_{j \in \mathbb{N}}\) that converges to \(x\) to conclude that the sequence \((x_n)_{n \in \mathbb{N}}\) itself converges to \(x\).

Let \((x_{n_j})_{j \in \mathbb{N}}\) be a subsequence of \((x_n)_{n \in \mathbb{N}}\). Since \(X\) is compact, the sequence \((x_{n_j})_{j \in \mathbb{N}}\) has a subsequence \((x_{n_{jk}})_{k \in \mathbb{N}}\) that converges to some \(\tilde{x} \in X\), and it remains to show that \(\tilde{x} = x\). By continuity of \(f\), we have \(f(x_{n_{jk}}) \to f(\tilde{x})\) as \(k \to \infty\). But since also \(f(x_{n_{jk}}) = y_{n_{jk}} \to y = f(x)\), the fact that limits are unique implies that \(f(x) = f(\tilde{x})\). Since \(f\) is injective, we conclude that \(x = \tilde{x}\). This proves the desired result.

(d) First we prove this result with sequences. Suppose \(((x_n, y_n))_{n \in \mathbb{N}}\) is a sequence in \(X \times Y\). Since \(X\) is compact, the sequence \((x_n)_{n \in \mathbb{N}}\) has a convergent subsequence \((x_{n_j})_{j \in \mathbb{N}}\) with limit \(x \in X\). Next, using the fact that \(Y\) is compact, the sequence \((y_n)_{n \in \mathbb{N}}\) has a convergent subsequence \((y_{n_{jk}})_{k \in \mathbb{N}}\) with limit \(y \in Y\). Then, since also \(x_{n_{jk}} \to x\) as \(k \to \infty\), we have that \((x_{n_{jk}}, y_{n_{jk}}) \to (x, y)\) as \(k \to \infty\) in \(X \times Y\). Thus we have found that the sequence \(((x_n, y_n))_{n \in \mathbb{N}}\) has a convergent subsequence. We conclude that \(X \times Y\) is compact.

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We point out here that we used the fact a sequence \(((a_n, b_n))_{n \in \mathbb{N}}\) in the product \(X \times Y\) converges to \((a, b) \in X \times Y\) with respect to any of the equivalent metrics in Lemma 1 if and only if both \(a_n \to a\) in \(X\) and \(b_n \to b\) in \(Y\).

Finally, we give a proof using open covers. For this we use the following fact:

**Lemma 2.** Let \((X, d)\) and \((Y, \rho)\) be metric spaces and let \(X \times Y\) be equipped with any of the equivalent metrics from Lemma 1. Then \(U \subseteq X \times Y\) is open if and only if for every point \((x, y) \in U\) there exist open sets \(U_1 \subseteq X\) and \(U_2 \subseteq Y\) with \(x \in U_1, y \in U_2,\) and \(U_1 \times U_2 \subseteq U\).

We leave the proof of this lemma as an exercise.

Now let \((U_i)_{i \in I}\) be an open cover of \(X \times Y\). Fix \(y_0 \in Y\). Then for each \(x \in X\), there exists an \(i(x, y_0) \in I\) such that \((x, y_0) \in U_{i(x, y_0)}\). By Lemma 2 there exist open sets \(A_{x, y_0} \subseteq X\), \(B_{x, y_0} \subseteq Y\) respectfully containing \(x\) and \(y_0\), such that \(A_{x, y_0} \times B_{x, y_0} \subseteq U_{i(x, y_0)}\). Then the collection \((A_{x, y_0})_{x \in X}\) is an open cover of \(X\). Since \(X\) is compact, we can find a finite set \(F_{y_0} \subseteq X\) such that \((A_{x, y_0})_{x \in F_{y_0}}\) still covers \(X\).

Next, for each \(y \in Y\) we set

\[ V_y := \bigcap_{x \in F_y} B_{x, y}. \]

Since this is the intersection of finitely many open sets in \(Y\), it is an open set in \(Y\). Moreover, since \(y \in V_y\) for each \(y \in Y\), the collection \((V_y)_{y \in Y}\) is an open cover of \(Y\). Hence, since \(Y\) is compact, we can find a finite set \(G \subseteq Y\) such that \((V_y)_{y \in G}\) still covers \(Y\).

Finally, consider the finite subcollection \(\bigcup_{y \in G} (U_{i(x,y)})_{x \in F_y}\) of our original open cover. Note that this is again a cover of \(X \times Y\). Indeed, for each \((a, b) \in X \times Y\) there is a \(y \in G\) such that \(b \in V_y\). Per definition of \(V_y\), this means that \(b \in B_{x,y}\) for all \(x \in F_y\). Picking \(x \in F_y\) such that \(a \in A_{x,y}\), we conclude that \((a, b) \in A_{x,y} \times B_{x,y} \subseteq U_{i(x,y)}\), as desired. We conclude that any open cover of \(X \times Y\) has a finite subcover and thus, that \(X \times Y\) is compact.

* **Exercise 3:** Compact inclusions into \(C([0, 1])\)

For a function \(f : [0, 1] \to \mathbb{R}\) and \(\alpha \in (0, 1]\) we define

\[ |f|_{C^{0, \alpha}} := \sup_{x, y \in [0, 1], x \neq y} \frac{|f(x) - f(y)|}{|x - y|^{\alpha}}. \]

Moreover, we define the space of \(\alpha\)-Hölder continuous functions on \([0, 1]\) as

\[ C^{0, \alpha}([0, 1]) := \{ f : [0, 1] \to \mathbb{R} : |f|_{C^{0, \alpha}} < \infty \}. \]

We equip this space with the norm \(\|f\|_{C^{0, \alpha}} := \|f\|_\infty + |f|_{C^{0, \alpha}}\).

(a) Show that the ball \(B_{C^{0, \alpha}} := \{ f \in C^{0, \alpha}([0, 1]) : \|f\|_{C^{0, \alpha}} < 1 \}\) is relatively compact in \(C([0, 1])\), i.e., its closure with respect to \(\|\cdot\|_\infty\) is a compact subset of \(C([0, 1])\).

Next, define \(C^1([0, 1])\) as the set of those functions \(f \in C([0, 1])\) that are continuously differentiable in \((0, 1)\), and whose derivative \(f'\) extends continuously to \([0, 1]\). We equip this space with the norm \(\|f\|_{C^1} := \|f\|_\infty + \|f'\|_\infty\).

(b) Show that the ball \(B_{C^1} := \{ f \in C^1([0, 1]) : \|f\|_{C^1} < 1 \}\) is relatively compact in \(C([0, 1])\).

(Hint: show that \(C^1([0, 1]) \subseteq C^{0, 1}([0, 1])\).)
Solution:

(a) First note that \(|f|_{C^{0,\alpha}} < \infty\) implies that
\[
|f(x) - f(y)| \leq |f|_{C^{0,\alpha}} |x - y|^\alpha
\]
for all \(x, y \in [0, 1]\). In particular, one can deduce from this that any \(f \in C^{0,\alpha}([0, 1])\) lies in \(C([0, 1])\) and thus also \(B_{C^{0,\alpha}} \subseteq C([0, 1])\).

The set \(B_{C^{0,\alpha}}\) is relatively compact in \(C([0, 1])\) if its closure is compact. This is equivalent\(^1\) to the property that every sequence \((f_n)_{n \in \mathbb{N}} \subseteq B_{C^{0,\alpha}}\) has a convergent subsequence with respect to \(\| \cdot \|\) and \(\| \cdot \|_{C^{0,\alpha}}\).

Let \((f_n)_{n \in \mathbb{N}}\) be such a sequence. We will use the Arzelà-Ascoli Theorem. Note that since
\[
\|f_n\|_\infty + |f|_{C^{0,\alpha}} = \|f_n\|_{C^{0,\alpha}} < 1 \quad \text{for all } n \in \mathbb{N},
\]
we have the two estimates
\[
\sup_{n \in \mathbb{N}} \|f_n\|_\infty \leq 1, \quad \sup_{n \in \mathbb{N}} |f_n(x) - f_n(y)| \leq |x - y|^\alpha \quad \text{for all } x, y \in [0, 1].
\]

The first estimate implies that \((f_n)_{n \in \mathbb{N}}\) is uniformly bounded. The second estimate implies that the sequence \((f_n)_{n \in \mathbb{N}}\) is uniformly equicontinuous: for \(\varepsilon > 0\), set \(\delta := \varepsilon^{\frac{1}{\alpha}} > 0\). Then if \(|x - y| < \delta\), we have
\[
|f_n(x) - f_n(y)| \leq |x - y|^{\alpha} < \delta^\alpha = \varepsilon
\]
for all \(n \in \mathbb{N}\).

Since the conditions of the Arzelà-Ascoli Theorem are met, we conclude that \((f_n)_{n \in \mathbb{N}}\) has a uniformly convergent subsequence. This was precisely what we had to show to conclude that \(B_{C^{0,\alpha}}\) is relatively compact in \(C([0, 1])\). The assertion follows.

(b) Let \(f \in C^1([0, 1])\) and let \(x, y \in [0, 1]\) be distinct points. Then it follows from the Mean Value Theorem that there exists a \(t\) between \(x\) and \(y\) such that \(f(x) - f(y) = f'(t)(x - y)\). In particular, it follows that
\[
|f(x) - f(y)| = |f'(t)||x - y| \leq \|f'\|_\infty |x - y|.
\]

Alternatively, one can use the Fundamental Theorem of Calculus to conclude this estimate as follows:
\[
|f(x) - f(y)| = \left| \int_y^x f'(t) \, dt \right| \leq \int_y^x |f'(t)| \, dt \leq \|f'\|_\infty |x - y|
\]
for \(x > y\). Either way, this implies that \(f \in C^{0,1}([0, 1])\) with \(|f|_{C^{0,1}} \leq \|f'\|_\infty\). Thus, we may also conclude that
\[
\|f\|_{C^{0,1}} = \|f\|_\infty + |f|_{C^{0,1}} \leq \|f\|_\infty + \|f'\|_\infty = \|f\|_{C^1}. \quad (2)
\]

Now pick \(f \in B_{C^1}\). Then it follows from (2) that also \(f \in B_{C^{0,1}}\). Thus, we conclude that \(B_{C^1} \subseteq B_{C^{0,1}}\). We have shown in part (a) that the letter set is relatively compact in \(C([0, 1])\). Thus, it follows from Exercise 2(b) that \(B_{C^1}\) is also relatively compact in \(C([0, 1])\). This proves the result.

* Exercise 4: Compactness of subsets of \(\ell^p\)

\(^1\)Check this!

— Turn the page! —
(a) Let $p \in [1, \infty)$. Suppose a bounded set $A \subseteq \ell^p$ satisfies the property that

$$\lim_{J \to \infty} \sup_{x \in A} \sum_{j=J+1}^{\infty} |x_j|^p = 0.$$ 

Prove that $A$ is totally bounded.

(b) Show that the set $A := \{ x \in \ell^1 : |x_j| \leq 2^{-j} \text{ for all } j \in \mathbb{N} \}$ is compact.

**Solution:**

(a) First we will show that the set

$$B_J := \{ \tilde{x} \in \ell^p : \tilde{x}_j = 0 \text{ for all } j > J \text{ and } \exists x \in A \text{ such that } \tilde{x}_j = x_j \text{ for } j \in \{1, \ldots, J\} \}$$

is totally bounded for any $J \in \mathbb{N}$. The idea is that $B_J$ can be viewed as a subset of $\mathbb{R}^J$. Since we assumed that $A$ is bounded, so is $B_J$. Thus, we can use Exercise 1(c) to conclude that $B_J$ is totally bounded. We will make this idea precise at the end of this solution. For now, assume that we have indeed shown that $B_J$ is totally bounded.

Let $\varepsilon > 0$. By the assumption we can pick a $J \in \mathbb{N}$ such that

$$\sup_{x \in A} \sum_{j=J+1}^{\infty} |x_j|^p < \left( \frac{\varepsilon}{2} \right)^p.$$

We claim that for each $x \in A$ there is an $\tilde{x} \in B_J$ such that $\|x - \tilde{x}\|_p < \frac{\varepsilon}{2}$.

Indeed, let $x = (x_j)_{j \in \mathbb{N}} \in A$. Then we define a new sequence $\tilde{x} = (\tilde{x}_j)_{j \in \mathbb{N}}$ by setting $\tilde{x}_j = x_j$ if $j \in \{1, \ldots, J\}$ and $\tilde{x}_j = 0$ if $j > J$. Note that then $\tilde{x} \in B_J$. Moreover, we have

$$\|x - \tilde{x}\|_p = \left( \sum_{j=J+1}^{\infty} |x_j|^p \right)^{\frac{1}{p}} \leq \left( \sup_{x \in A} \sum_{j=J+1}^{\infty} |x_j|^p \right)^{\frac{1}{p}} < \frac{\varepsilon}{2}.$$

This proves the claim.

Finally, since $B_J$ is totally bounded, we can find a finite collection of points $y^{(1)}, \ldots, y^{(N)} \in B_J$ such that

$$B_J \subseteq \bigcup_{n=1}^{N} B(y^{(n)}, \frac{\varepsilon}{2}).$$

(3)

If we can now show that $A \subseteq \bigcup_{n=1}^{N} B(y^{(n)}, \varepsilon)$, then we may conclude that $A$ is totally bounded.

To prove this inclusion, let $x \in A$. By our claim, we can find an $\tilde{x} \in B_J$ such that

$$\|x - \tilde{x}\|_p < \frac{\varepsilon}{2}.$$

(4)

But then by (3), we can find an $n \in \{1, \ldots, N\}$ such that $\tilde{x} \in B(y^{(n)}, \frac{\varepsilon}{2})$. Thus, by combining this fact with (4) and the triangle inequality, we obtain

$$\|x - y^{(n)}\|_p \leq \|x - \tilde{x}\|_p + \|\tilde{x} - y^{(n)}\|_p < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus indeed $x \in B(y^{(n)}, \varepsilon)$ and we may conclude that $A \subseteq \bigcup_{n=1}^{N} B(y^{(n)}, \varepsilon)$, proving the result.
Finally, we will explain in more detail why $B_J$ is totally bounded. On $\mathbb{R}^J$, we define the $p$-norm $\|x\|_{p,J} := \left( \sum_{j=1}^J |x_j|^p \right)^{\frac{1}{p}}$. Note that now $\| \cdot \|_{2,J}$ is just the usual Euclidean norm on $\mathbb{R}^J$. The map

$$\phi : B_J \to \{ \tilde{x} \in \mathbb{R}^J : \exists x \in A \text{ such that } \tilde{x}_j = x_j \text{ for } j \in \{1, \ldots, J\} \},$$

$$(\tilde{x}_j)_{j \in \mathbb{N}} \mapsto (\tilde{x}_1, \ldots, \tilde{x}_J)$$

is bijective, and moreover, $\|\phi(\tilde{x})\|_{p,J} = \|\tilde{x}\|_p$ for all $\tilde{x} \in B_J$.

Now let $\tilde{x} \in \phi(B_J)$ and pick $x \in A$ with $x_j = \tilde{x}_j$ for $j \in \{1, \ldots, J\}$. Then we have

$$\|\phi(\tilde{x})\|_{p,J} = \|\tilde{x}\|_p = \left( \sum_{j=1}^J |x_j|^p \right)^{\frac{1}{p}} \leq \|x\|_p \leq \sup_{x \in A} \|x\|_p < \infty,$$

where we use the fact that $A$ is bounded. We conclude that $\phi(B_J)$ is bounded in $\mathbb{R}^J$ with respect to the norm $\| \cdot \|_{p,J}$. Thus, since subsets of $\mathbb{R}^J$ are bounded if and only if they are totally bounded with respect to any norm$^2$, $\phi(B_J)$ is also totally bounded with respect to $\| \cdot \|_{p,J}$. Finally, let $\varepsilon > 0$ and $\tilde{x}^{(1)}, \ldots, \tilde{x}^{(N)} \in \phi(B_J)$ such that the union of the balls $(B_{\|\cdot\|_p}(\tilde{x}^{(n)}, \varepsilon))_{n=1}^N$ cover $\phi(B_J)$. Then the union of the balls $(B_{\|\cdot\|_p}(\phi^{-1}(\tilde{x}^{(n)}), \varepsilon))_{n=1}^N$ cover $B_J$. We conclude that $B_J$ is totally bounded, as desired.

(b) From Exercise 2(c) of Exercise sheet 0 we know that $A$ is closed. Since $A$ is a closed subset of the complete space $\ell^1$, we conclude that $A$ is also complete. To conclude that $A$ is compact, it remains to check that $A$ is totally bounded. For this we use part (a).

Note that for each $x \in A$ we have

$$\sum_{j=J+1}^\infty |x_j| \leq \sum_{j=J+1}^\infty 2^{-j} = \sum_{j=0}^\infty (2^{-1})^{J+j+1} = 2^{-J}.$$ 

Hence,

$$\sup_{x \in A} \sum_{j=J+1}^\infty |x_j|^p \leq 2^{-J} \to 0, \quad \text{as } J \to \infty.$$

Thus, the condition of part (a) holds for $A$, and we may conclude that $A$ is totally bounded. The assertion follows.

\[\text{http://www.math.kit.edu/iana3/edu/fa2019w/de}\]

\[\text{We are using a particular fact about equivalent norms here. In Exercise 1 we have shown that the notions of boundedness and totally boundedness need not be preserved after switching to equivalent metrics. However, these notions are preserved when switching to equivalent norms! Moreover, we note the remarkable fact that any two norms on } \mathbb{R}^n \text{ are equivalent (check this if you haven’t seen this!).}\]