Remark. The exercises marked with a * can be handed in for correction in the “Funktionalanalysis” box in the atrium of building 20.30 at the latest at 14:00 on the day of the exercise class next week.

Exercise 1: The operator norm
Let \( L(X,Y) \) be the space of bounded linear operators between two normed spaces \( X \neq \{0\} \) and \( Y \), equipped with the operator norm \( \|T\| = \sup_{x \in X, x \neq 0} \frac{\|Tx\|}{\|x\|} \).

(a) Prove the equalities
\[
\|T\| = \inf \{ C \geq 0 : \|Tx\| \leq C\|x\| \text{ for all } x \in X \} = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\|\leq 1} \frac{\|Tx\|}{\|x\|}.
\]

(b) Prove that \( L(X,Y) \) is a vector space and prove that \( \| \cdot \| \) is a norm on \( L(X,Y) \).

* Exercise 2: Multiplication operators on \( \ell^p \)
Let \( y = (y_j)_{j \in \mathbb{N}} \subseteq \mathbb{K} \) be a fixed sequence. For each sequence \( x = (x_j)_{j \in \mathbb{N}} \subseteq \mathbb{K} \) we define a new sequence
\[
T_yx := (y_jx_j)_{j \in \mathbb{N}}.
\]
Note that \( T_y \) defines a linear operator on the space of all sequences in \( \mathbb{K} \).

Let \( p \in [1, \infty] \). Show that the following are equivalent:

(i) \( T_y(\ell^p) \subseteq \ell^p \) and \( T_y : \ell^p \to \ell^p \) is a bounded operator.

(ii) \( y \in \ell^\infty \);

Moreover, show that in this case we have \( \|T_y\| = \|y\|_\infty \).

* Exercise 3: Integral operators
Let \( k \) be a continuous function \( k : [0,1] \times [0,1] \to \mathbb{R} \). For each \( f \in C([0,1]) \) we define a function \( Tf : [0,1] \to \mathbb{R} \) by
\[
Tf(t) := \int_0^1 k(t,s)f(s) \, ds.
\]

(a) Show that \( Tf \in C([0,1]) \) for \( f \in C([0,1]) \) and, moreover, show that \( T : C([0,1]) \to C([0,1]) \) is a linear operator.
(b) Show that
\[ \|T\| = \sup_{t \in [0,1]} \int_{0}^{1} |k(t, s)| \, ds. \]

**Exercise 4: A reverse estimate**
Let \( X, Y \) be normed spaces and let \( T \in L(X, Y) \). Suppose there is a \( c > 0 \) such that
\[ \|Tx\| \geq c\|x\| \quad \text{for all } x \in X. \]

(a) Show that if \( X \) is a Banach space, then the range \( R(T) \) of \( T \) is closed in \( Y \).

(b) Show that \( T \) is injective and show that the operator \( T^{-1} : R(T) \to X \) is bounded with \( \|T^{-1}\| \leq c^{-1} \).

**Exercise 5: Invertible operators**
Let \( X \) be a Banach space and denote the set of invertible operators in \( L(X) \) by \( U(L(X)) \).

(a) Suppose \( T \in L(X) \) with \( \|T\| < 1 \). Prove that \( \text{Id} - T \in U(L(X)) \) with
\[ (\text{Id} - T)^{-1} = \sum_{n=0}^{\infty} T^n, \]
where the series converges in \( L(X) \).

(b) Prove that \( U(L(X)) \) is an open subset of \( L(X) \).

\( \text{(Hint: first show that if } S \in L(X) \text{ is close enough to } T \in U(L(X)), \text{ then } ST^{-1} \text{ is invertible.)} \)

**Exercise 6: Projections induced by direct sums**
Let \( X \) be a normed space and suppose that \( X \) decomposes as a direct sum \( X = X_1 \oplus X_2 \). We define a map \( P : X \to X \) by \( Px := x_1 \), where \( x = x_1 + x_2 \) is the unique decomposition of \( x \in X \) with \( x_1 \in X_1 \) and \( x_2 \in X_2 \).

Prove that \( P \) is the unique linear projection satisfying \( R(P) = X_1 \) and \( N(P) = X_2 \).