Functional Analysis

Solutions to exercise sheet 4

**Exercise 1: The operator norm**

Let $L(X,Y)$ be the space of bounded linear operators between two normed spaces $X \neq \{0\}$ and $Y$, equipped with the operator norm $\|T\| = \sup_{\|x\| \neq 0} \frac{\|Tx\|}{\|x\|}$.

(a) Prove the equalities

\[
\|T\| = \inf\{C \geq 0 : \|Tx\| \leq C\|x\| \text{ for all } x \in X\} = \sup_{\|x\| = 1} \|Tx\| = \sup_{\|x\| \leq 1} \|Tx\|.
\]

(b) Prove that $L(X,Y)$ is a vector space and prove that $\| \cdot \|$ is a norm on $L(X,Y)$.

**Solution:**

(a) Note that the estimate

\[
\sup_{x \in X} \|Tx\| \leq \sup_{\|x\| = 1} \|Tx\|
\]

is clear, since we are taking suprema of the same quantity over a larger set. Now let $x \in X$ with $\|x\| \leq 1$. Then if $x = 0$, we have $\|Tx\| = 0 \leq \|T\|$. Otherwise,

\[
\|Tx\| \leq \|Tx\| \leq \|T\|.
\]

Thus, taking a supremum over all $x \in X$ with $\|x\| \leq 1$, we conclude that $\sup_{\|x\| \leq 1} \|Tx\| \leq \|T\|$.

It remains to show that

\[
\|T\| \leq \inf\{C \geq 0 : \|Tx\| \leq C\|x\| \text{ for all } x \in X\} \leq \sup_{\|x\| = 1} \|Tx\|
\]

to conclude the equalities.

For the first inequality, suppose that $C \geq 0$ satisfies the property that $\|Tx\| \leq C\|x\|$ for all $x \in X$. Then for all $x \in X \setminus \{0\}$ we have

\[
\frac{\|Tx\|}{\|x\|} \leq C.
\]

Taking a supremum over all such $x$ yields $\|T\| \leq C$. Then, taking an infimum over all such $C \geq 0$ proves the first inequality.
For the second inequality, note that if $y \in X \setminus \{0\}$, then
\[
\|Ty\| = \|T\left(\frac{y}{\|y\|}\right)\|y\| \leq \sup_{x \in X} \|Tx\|\|y\|.
\]

Since the inequality $\|Ty\| \leq \sup_{x \in X} \|Tx\|\|y\|$ also holds for $y = 0$, we conclude that
\[
\inf\{C \geq 0 : \|Tx\| \leq C\|x\| \text{ for all } x \in X\} \leq \sup_{x \in X} \|Tx\|,
\]
as desired. The assertion follows.

(b) We only prove that if $T, S \in L(X, Y)$, then also $T + S \in L(X, Y)$ with $\|T + S\| \leq \|T\| + \|S\|$.

Let $x \in X$. Then
\[
\|(T + S)x\| = \|Tx + Sx\| \leq \|Tx\| + \|Sx\| \leq \|T\||\|x\| + \|S\||\|x\| = (\|T\| + \|S\|)\|x\|.
\]
This proves that $T + S \in L(X, Y)$. Since, by the second characterization of part (a), the norm $\|T + S\|$ is given by the smallest constant $C \geq 0$ such that $\|(T + S)x\| \leq C\|x\|$ for all $x \in X$, it follows from (1) that $\|T + S\| \leq \|T\| + \|S\|$. The result follows.

**Exercise 2: Multiplication operators on $\ell^p$**

Let $y = (y_j)_{j \in \mathbb{N}} \subseteq \mathbb{K}$ be a fixed sequence. For each sequence $x = (x_j)_{j \in \mathbb{N}} \subseteq \mathbb{K}$ we define a new sequence
\[
T_y x := (y_j x_j)_{j \in \mathbb{N}}.
\]
Note that $T_y$ defines a linear operator on the space of all sequences in $\mathbb{K}$.

Let $p \in [1, \infty]$. Show that the following are equivalent:

(i) $T_y(\ell^p) \subseteq \ell^p$ and $T_y : \ell^p \to \ell^p$ is a bounded operator.

(ii) $y \in \ell^\infty$.

Moreover, show that in this case we have $\|T_y\| = \|y\|_\infty$.

**Solution:** Fix $p \in [1, \infty]$. First we prove the implication (ii)$\Rightarrow$(i). If $y \in \ell^\infty$, then
\[
\|T_y x\|_p = \|(y_j x_j)_{j \in \mathbb{N}}\|_p \leq \|y\|_\infty \|x\|_p
\]
which is finite whenever $x \in \ell^p$. Thus indeed, $T_y(\ell^p) \subseteq \ell^p$. Moreover, we point out that this estimate implies that $\|T_y\| \leq \|y\|_\infty$.

Next we prove that (i)$\Rightarrow$(ii). By (i), there is a constant $C \geq 0$ such that
\[
\|T_y x\|_p \leq C\|x\|_p
\]
for all $x \in \ell^p$.

For $j \in \mathbb{N}$ we consider the sequence $e_j$, where $e_j$ is the sequence that is 0 everywhere except in its $j$-th place, where it is equal to 1. Then certainly $e_j \in \ell^p$ with $\|e_j\|_p = 1$. Note that $T_y e_j$ is the sequence that is 0 everywhere except in its $j$-th place, where it is equal to $y_j$. Then, by (2), we have
\[
\|y_j\| = \|T_y e_j\|_p \leq C\|e_j\| = C.
\]
Thus, \( y \in \ell^\infty \) with \( \|y\|_\infty \leq C \), proving the assertion. Furthermore, by taking an infimum over all possible \( C \geq 0 \) such that (2) holds, we conclude that \( \|y\|_\infty \leq \|T_y\| \). Combining this with our estimate from the other implication we conclude that \( \|T_y\| = \|y\|_\infty \). This proves the desired result.

**Exercise 3: Integral operators**

Let \( k \) be a continuous function \( k : [0, 1] \times [0, 1] \to \mathbb{R} \). For each \( f \in C([0, 1]) \) we define a function \( Tf : [0, 1] \to \mathbb{R} \) by

\[
Tf(t) := \int_0^1 k(t, s)f(s) \, ds.
\]

(a) Show that \( Tf \in C([0, 1]) \) for \( f \in C([0, 1]) \) and, moreover, show that \( T : C([0, 1]) \to C([0, 1]) \) is a linear operator.

(b) Show that

\[
\|T\| = \sup_{t \in [0, 1]} \int_0^1 |k(t, s)| \, ds.
\]

**Solution:**

(a) Let \( f \in C([0, 1]) \). We will show that \( Tf \) is in fact a uniformly continuous function. We set

\[
\|f\|_1 := \int_0^1 |f(s)| \, ds.
\]

Note that this constant is finite, since

\[
\|f\|_1 = \int_0^1 |f(s)| \, ds \leq \|f\|_\infty \int_0^1 \, ds = \|f\|_\infty < \infty.
\]

Next, let \( \varepsilon > 0 \). Since \( k \) is a continuous function on the compact set \([0, 1] \times [0, 1]\), it is actually uniformly continuous. Thus, we can choose \( \delta > 0 \) such that whenever \( |(t, s) - (t', s')| < \delta \), we have \( |k(t, s) - k(t', s')| < \frac{\varepsilon}{\|f\|_1} \).

If \( t, t' \in [0, 1] \) satisfy \( |t - t'| < \delta \), then for each \( s \in [0, 1] \) we have \( |(t, s) - (t', s)| = |(t - t', 0)| = |t - t'| < \delta \). Hence,

\[
|Tf(t) - Tf(t')| = \left| \int_0^1 (k(t, s) - k(t', s))f(s) \, ds \right| \leq \int_0^1 |k(t, s) - k(t', s)||f(s)| \, ds
\]

\[
\leq \frac{\varepsilon}{1 + \|f\|_1} \int_0^1 |f(s)| \, ds = \frac{\varepsilon \|f\|_1}{1 + \|f\|_1} < \varepsilon.
\]

Thus, \( Tf \) is uniformly continuous. In particular \( Tf \in C([0, 1]) \).

We point out here that the only property we needed of \( f \) is that it is integrable over \([0, 1]\) to conclude the continuity of \( Tf \).

The linearity of \( T \) follows from the linearity of the integral.

(b) Set \( A := \sup_{t \in [0, 1]} \int_0^1 |k(t, s)| \, ds \). We will show that \( \|T\| \leq A \) and \( A \leq \|T\| \) to conclude that \( \|T\| = A \).

For the first inequality, note that for all \( t \in [0, 1] \) we have

\[
|Tf(t)| \leq \int_0^1 |k(t, s)||f(s)| \, ds \leq \int_0^1 |k(t, s)| \, ds \|f\|_\infty \leq \sup_{t \in [0, 1]} \int_0^1 |k(t, s)| \, ds \|f\|_\infty = A\|f\|_\infty.
\]

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Taking a supremum over all \( t \in [0, 1] \) yields \( \|Tf\|_\infty \leq A\|f\|_\infty \). Thus, \( \|T\| \leq A \), as desired.

For the converse inequality, note that since \( k \) is continuous, so is \( |k| \). Hence, by the same argument as in part (a), the map \( t \to \int_0^1 |k(t, s)| \, ds \) for \( t \in [0, 1] \) is also continuous. Since \([0, 1]\) is compact, this means that there is a \( t_0 \in [0, 1] \) where the supremum of this function is attained, i.e., \( A = \sup_{t \in [0, 1]} \int_0^1 |k(t, s)| \, ds = \int_0^1 |k(t_0, s)| \, ds \). Now for \( n \in \mathbb{N} \) we define

\[
    f_n : [0, 1] \to \mathbb{R}, \quad f_n(s) := \frac{k(t_0, s)}{|k(t_0, s)| + \frac{1}{n}}.
\]

Note that then \( \|f_n\|_\infty \leq 1 \) so that

\[
    |Tf_n(t_0)| \leq \|Tf_n\|_\infty \leq \|f_n\|_\infty \|T\| \leq \|T\|. \tag{3}
\]

But

\[
    |Tf_n(t_0)| = \int_0^1 \frac{|k(t_0, s)|^2}{|k(t_0, s)| + \frac{1}{n}} \, ds = \int_0^1 \frac{|k(t_0, s)|}{|k(t_0, s)| + \frac{1}{n}} \left(1 - \frac{\frac{1}{n}}{|k(t_0, s)| + \frac{1}{n}}\right) \, ds
\]

\[
= \int_0^1 \frac{|k(t_0, s)|}{|k(t_0, s)| + \frac{1}{n}} \, ds - \frac{1}{n} \int_0^1 \frac{|k(t_0, s)|}{|k(t_0, s)| + \frac{1}{n}} \, ds \to \int_0^1 |k(t_0, s)| \, ds = A
\]

as \( n \to \infty \). Thus, letting \( n \to \infty \) in (3), we conclude that \( A \leq \|T\| \). Thus, we have \( \|T\| = A \), proving the result.

**Exercise 4: A reverse estimate**

Let \( X, Y \) be normed spaces and let \( T \in L(X, Y) \). Suppose there is a \( c > 0 \) such that

\[
    \|Tx\| \geq c\|x\| \quad \text{for all } x \in X.
\]

(a) Show that if \( X \) is a Banach space, then the range \( R(T) \) of \( T \) is closed in \( Y \).

(b) Show that \( T \) is injective and show that the operator \( T^{-1} : R(T) \to X \) is bounded with \( \|T^{-1}\| \leq c^{-1} \).

**Solution:**

(a) Suppose \( (y_n)_{n \in \mathbb{N}} \) is a sequence in \( R(T) \) with limit \( y \in Y \). To conclude that \( R(T) \) is closed, we need to show that \( y \in R(T) \). For each \( n \in \mathbb{N} \), choose \( x_n \in X \) such that \( Tx_n = y_n \). Then we have

\[
    \|x_n - x_m\| \leq \|T(x_n - x_m)\| = \|y_n - y_m\| \to 0, \quad \text{as } n, m \to \infty.
\]

Thus, \( (x_n)_{n \in \mathbb{N}} \) is a Cauchy sequence in \( X \). Since \( X \) is Banach, this sequence converges to some limit \( x \in X \). By continuity of \( T \), this implies that \( y_n = Tx_n \to Tx \) in \( Y \) as \( n \to \infty \). Since also \( y_n \to y \) in \( Y \) as \( n \to \infty \), uniqueness of limits implies that \( y = Tx \in R(T) \). The assertion follows.

(b) For \( x \in X \) we note that if \( Tx = 0 \), then \( 0 = \|Tx\| \geq c\|x\| \), implying that also \( x = 0 \). Thus, \( T \) is injective and is thus invertible on its range. Let \( y \in R(T) \) and set \( x := T^{-1}y \) so that \( Tx = y \). Then

\[
    c^{-1}\|y\| = c^{-1}\|Tx\| \geq \|x\| = \|T^{-1}y\|.
\]

Hence, \( \|T^{-1}\| \leq c^{-1} \).

**Exercise 5: Invertible operators**

Let \( X \) be a Banach space and denote the set of invertible operators in \( L(X) \) by \( U(L(X)) \).

(a) Suppose $T \in L(X)$ with $\|T\| < 1$. Prove that $\text{Id} - T \in U(L(X))$ with

$$(\text{Id} - T)^{-1} = \sum_{n=0}^{\infty} T^n,$$

where the series converges in $L(X)$.

(b) Prove that $U(L(X))$ is an open subset of $L(X)$.

(*Hint: first show that if $S \in L(X)$ is close enough to $T \in U(L(X))$, then $ST^{-1}$ is invertible.*)

**Solution:**

(a) We will use the following general result:

**Lemma 1.** Let $S, T \in L(X)$. Then $ST \in L(X)$, where $ST$ is the composition of $S$ with $T$. Moreover, we have

$$\|ST\| \leq \|S\|\|T\|.$$  

In particular, the maps $L(X) \to L(X)$, $T \mapsto ST$ and $L(X) \to L(X)$, $T \mapsto TS$ are continuous.

**Proof.** For any $x \in X$ we have $\|STx\| \leq \|S\|\|Tx\| \leq \|S\|\|T\|\|x\|$. Thus, $ST \in L(X)$ with $\|ST\| \leq \|S\|\|T\|$, as desired.

For the continuity property, we prove the first one. The second one follows from an analogous argument. To prove that $T \mapsto ST$ is continuous, let $(T_n)_{n \in \mathbb{N}}$ be a sequence in $L(X)$ that converges to $T \in L(X)$. Then we have

$$\|ST_n - ST\| = \|S(T_n - T)\| \leq \|S\|\|T_n - T\| \to 0 \quad \text{as } n \to \infty$$

so that $ST_n \to ST$ in $L(X)$ as $n \to \infty$. This proves the result.

Note that since $\|T\| < 1$, the geometric series $\sum_{n=0}^{\infty} \|T\|^n$ converges with value $\frac{1}{1 - \|T\|}$. Setting $T^0 := \text{Id}$, we note that it follows from an iterated usage of the inequality in Lemma 1 that $\|T^n\| \leq \|T\|^n$. Hence,

$$\sum_{n=0}^{\infty} \|T^n\| \leq \sum_{n=0}^{\infty} \|T\|^n = \frac{1}{1 - \|T\|} < \infty.$$  

Since $X$ is a Banach space, $L(X)$ is also a Banach space. Since absolutely convergent series in Banach spaces converge, this means that there is an operator $S \in L(X)$ with

$$\sum_{n=0}^{\infty} T^n = S,$$

where the series should be interpreted as a series in $L(X)$. We will show that $S = (\text{Id} - T)^{-1}$.

Write $R_N := \sum_{n=0}^{N} T^n$ so that $R_N \to S$ in $L(X)$ as $N \to \infty$. Note that then

$$TR_N = \sum_{n=1}^{N+1} T^n = R_{N+1} - \text{Id}$$

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Then it follows from Lemma 1 that

\[(\text{Id} - T)S = \lim_{N \to \infty} (\text{Id} - T)R_N = \lim_{N \to \infty} R_N - TR_N = \lim_{N \to \infty} R_N - R_{N+1} + \text{Id} = S - S + \text{Id} = \text{Id},\]

where the limits are in \(L(X)\). We conclude that \((\text{Id} - T)S = \text{Id}\).

Analogously, we can also show that \(S(\text{Id} - T) = \text{Id}\). Thus, \(\text{Id} - T\) is invertible with inverse \(S\). This proves the result.

(b) Let \(T \in U(L(X))\). We will show that \(B(T, \|T^{-1}\|^{-1}) \subseteq U(L(X))\), which would prove that \(U(L(X))\) is open.

Let \(S \in B(T, \|T^{-1}\|^{-1})\). Then, by Lemma 1, we have

\[\| \text{Id} - ST^{-1} \| = \|(T - S)T^{-1} \| \leq \|T - S\|\|T^{-1} \| < \frac{\|T^{-1} \|}{\|T^{-1} \|} = 1.\]

Thus, by part (a), \(ST^{-1} = \text{Id} - (\text{Id} - ST^{-1})\) is invertible. Let \(R := (ST^{-1})^{-1} \in L(X)\). Then we claim that \(T^{-1}R = S^{-1}\). Indeed, by Lemma 1 we have \(T^{-1}R \in L(X)\). Moreover, we have

\[T^{-1}RS = T^{-1}R(ST^{-1})T = T^{-1}\text{Id} \text{T} = \text{Id}\]

and

\[ST^{-1}R = (ST^{-1})R = \text{Id}.\]

Thus, \(T^{-1}R\) is both a left and a right inverse of \(S\), proving that \(S \in U(L(X))\) with inverse \(T^{-1}R\). Thus we have shown that \(B(T, \|T^{-1}\|^{-1}) \subseteq U(L(X))\), proving the assertion.

**Exercise 6: Projections induced by direct sums**

Let \(X\) be a normed space and suppose that \(X\) decomposes as a direct sum \(X = X_1 \oplus X_2\). We define a map \(P : X \to X\) by \(Px := x_1\), where \(x = x_1 + x_2\) is the unique decomposition of \(x \in X\) with \(x_1 \in X_1\) and \(x_2 \in X_2\).

Prove that \(P\) is the unique linear projection satisfying \(R(P) = X_1\) and \(N(P) = X_2\).

**Solution:** First we show that \(P\) is well-defined. We have defined a direct sum as follows: for two subspaces \(X_1, X_2 \subseteq X\) we write \(X = X_1 \oplus X_2\) if each \(x \in X\) can be written as \(x = x_1 + x_2\) for \(x_1 \in X_1\) and \(x_2 \in X_2\), and \(X_1 \cap X_2 = \{0\}\). This latter condition implies that the decomposition \(x = x_1 + x_2\) is indeed unique, for if also \(x = x_1' + x_2'\) for some \(x_1' \in X_1, x_2' \in X_2\), then \(0 = x - x = (x_1 - x_1') + (x_2 - x_2')\) and thus \(x_1 - x_1' = x_2' - x_2 \in X_1 \cap X_2 = \{0\}\). Hence \(x_1 = x_1'\) and \(x_2 = x_2'\), proving that the decomposition of \(x\) is unique, as asserted\(^1\).

Since decompositions are unique, this means that \(P\) is indeed well-defined. Moreover, it is a projection. Indeed, if \(x \in X\), and \(x = x_1 + x_2\) for \(x_1 \in X_1, x_2 \in X_2\), then \(Px = x_1\). Applying \(P\) again, noting that the unique decomposition of \(x_1\) is given by \(x_1 = x_1 + 0\), we find that \(Px_1 = x_1\). Thus,

\[P^2x = Px_1 = x_1 = Px,\]

showing that \(P\) satisfies \(P^2 = P\).

\(^1\)Of course, the converse is also true: if any \(x \in X\) can be uniquely decomposed as \(x = x_1 + x_2\) for \(x_1 \in X_1\) and \(x_2 \in X_2\), then \(X_1 \cap X_2 = \{0\}\).
To prove that $P$ is linear, we note that if $x, y \in X$, and their respective decompositions are given by $x = x_1 + x_2$, $y = y_1 + y_2$, then the decomposition of $x + \lambda y$ for $\lambda \in \mathbb{K}$ is given by

$$x + \lambda y = (x_1 + \lambda y_1) + (x_2 + \lambda y_2).$$

Hence, $P(x + \lambda y) = x_1 + \lambda y_1 = Px + \lambda Py$, as desired.

Finally, we show that $R(P) = X_1$ and $N(P) = X_2$. For the former, note that $R(P) \subseteq X_1$ is clear. For the converse inclusion, let $x \in X_1$. Then the decomposition of $x$ is given by $x = x + 0$, so $x = Px \in R(P)$. Thus $R(P) = X_1$.

To prove $N(P) = X_2$, note that if $x \in X_2$, then its decomposition is given by $x = 0 + x$ so that $Px = 0$. Hence, $X_2 \subseteq N(P)$. For the converse inclusion, let $x \in N(P)$. Writing $x = x_1 + x_2$, we have $0 = Px = x_1$. Hence, $x = x_2 \in X_2$. We conclude that $N(P) = X_2$ and the assertion follows.

The fact that $P$ is the unique projection satisfying $R(P) = X_1$ and $N(P) = X_2$ follows from a general result about projections.

**Lemma 2.** Suppose two projections $P_1$ and $P_2$ on $X$ satisfy $R(P_1) = R(P_2)$ and $N(P_1) = N(P_2)$. Then $P_1 = P_2$.

**Proof.** First we will show that for any $y \in R(P_1)$ we have $P_1y = y$. Indeed, if $y \in R(P_1)$ then there is an $x \in X$ such that $y = P_1x$. Thus, $P_1y = P_1^2x = P_1x = y$, as desired.

Next, let $x \in X$. Note that since $P_1x \in R(P_1)$ and $P_2x \in R(P_2) = R(P_1)$, we also have $P_1x - P_2x \in R(P_1)$. Hence, by our previous observation, $P_1(P_1x - P_2x) = P_1x - P_2x$. But then

$$P_1x - P_2x = P_1(P_1x - P_2x) = P_1^2x - P_1P_2x = P_1x - P_1P_2x = P_1(Id-P_2)x.$$

It remains to show that $P_1(Id-P_2)x = 0$ to conclude that $P_1x = P_2x$.

To prove this result, note that $P_2(Id-P_2)x = P_2x - P_2^2x = P_2x - P_2x = 0$. This implies that $(Id-P_2)x \in N(P_2)$. But since $N(P_1) = N(P_2)$, this means that also $P_1(Id-P_2)x = 0$. The assertion follows. \qed

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