Functional Analysis

Solutions to exercise sheet 5

Exercise 1: The norm on $\ell^\infty / c_0$.
Show that the quotient norm on $\ell^\infty / c_0$ satisfies the identity
$$\|\hat{x}\| = \limsup_{j \to \infty} |x_j|,$$
where $x = (x_j)_{j \in \mathbb{N}} \in \ell^\infty$, and where $\hat{x}$ denotes the class represented by $x$ in $\ell^\infty / c_0$.

Solution: Let $x \in \ell^\infty$ and for each $n \in \mathbb{N}$ we set $\pi_n x := (x_1, \ldots, x_n, 0, 0, \ldots) \in c_0$. Then,
$$\|\hat{x}\| = \inf_{y \in c_0} \|x - y\|_{\infty} \leq \inf_{n \in \mathbb{N}} \|x - \pi_n x\|_{\infty} = \inf_{n \in \mathbb{N}} \sup_{j \geq n} |x_j| = \limsup_{j \to \infty} |x_j|.$$
For the converse inequality, let $y \in c_0$. Then
$$\limsup_{j \to \infty} |x_j| \leq \limsup_{j \to \infty} (|x_j - y_j| + |y_j|) = \limsup_{j \to \infty} |x_j - y_j| \leq \|x - y\|_{\infty}.$$ Taking an infimum over all $y \in c_0$ yields $\limsup_{j \to \infty} |x_j| \leq \|\hat{x}\|$. Thus, $\|\hat{x}\| = \limsup_{j \to \infty} |x_j|$, as desired.

Exercise 2: Quotients of $C([0,1])$.
Let $0 \leq a \leq b \leq 1$ and set $[a,b] \subseteq [0,1]$.

(a) Show that $J := \{ f \in C[0,1] : f|_{[a,b]} = 0 \}$ is a closed subset of $C([0,1])$.

Let $f \in C([0,1])$ and let $\hat{f}$ denote the class represented by $f$ in $C([0,1])/J$.

(b) Show that $\hat{f}$ consists precisely of those functions that coincide with $f$ on $[a,b]$.

(c) Prove that $C([0,1])/J \cong C([a,b])$, i.e., prove that there is a continuous linear isomorphism between the two spaces with a continuous inverse.

Solution:

(a) Suppose $(f_n)_{n \in \mathbb{N}}$ is a sequence in $J$ that converges in $C([0,1])$ with limit $f \in C([0,1])$. To conclude that $J$ is closed, we need to show that $f \in J$. Note that since uniform convergence implies pointwise convergence, we have that for every $x \in [a,b]$,
$$f(x) = \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} f_n|_{[a,b]}(x) = 0.$$
Hence, $f|_{[a,b]} = 0$ and thus $f \in J$. The assertion follows.
(b) If \( g \) is in the class \( \hat{f} \), then \( g - f \in J \). Thus, for every \( x \in [a, b] \) we have
\[
g(x) - f(x) = (g - f)|_{[a,b]}(x) = 0.
\]
Hence, \( g(x) = f(x) \) for every \( x \in [a, b] \), as asserted.

(c) We define \( T : C([0,1])/J \to C([a,b]) \) by \( T \hat{f} := f|_{[a,b]} \). Note that this map is well-defined. Indeed, if \( g \in \hat{f} \), then by part (b) we have \( f|_{[a,b]} = g|_{[a,b]} \) so that the value of \( T \hat{f} \) does not depend on the representative chosen for \( \hat{f} \). We omit the proof of the linearity of \( T \), but we will prove that \( T \) satisfies the remaining necessary properties.

We check that \( T \) is bijective. To see that it is injective, suppose that \( T \hat{f} = 0 \) for some \( \hat{f} \in C([0,1])/J \). Then \( f|_{[a,b]} = 0 \). Since the 0 function also vanishes on \([a, b]\), both 0 and \( f \) are in the same class. Hence \( \hat{f} = 0 = 0 \), proving that \( T \) is injective.

To prove that \( T \) is surjective, let \( g \in C([a,b]) \). We define \( f : [0,1] \to \mathbb{R} \) by
\[
f(x) := \begin{cases} 
g(x) & \text{if } x \in [a, b], 
g(a) & \text{if } x \in [0,a], 
g(b) & \text{if } x \in [b,1]. 
\end{cases}
\]
(1)

Then \( f \in C([0,1]) \) and \( f \) coincides with \( g \) on \([a, b]\). Thus, \( T \hat{f} = f|_{[a,b]} = g \), proving that \( T \) is surjective. We conclude that \( T \) is bijective.

To prove that \( T \) is continuous, we will use a general result.

**Proposition 1.** Let \( X \) be a normed space and let \( V \) be a closed subspace of \( X \). Let \( Q : X \to X/V \) denote the quotient map and let \( Y \) be a normed space. Then an operator \( T : X/V \to Y \) is continuous if and only if \( T \circ Q : X \to Y \) is continuous.

For the proof we use the following lemma.

**Lemma 2.** If a set \( U \subseteq X \) is open, then \( Q(U) \) is open in \( X/V \).

**Proof.** To show that \( Q(U) \) is open, we will show that all of its points are interior points. Let \( \hat{x} \in Q(U) \) and pick \( x \in U \) such that \( Qx = \hat{x} \). Since \( U \) is open, there is an \( r > 0 \) such that \( B(x,r) \subseteq U \).

We claim that \( B_{X/V}(\hat{x},r) \subseteq Q(U) \), which would prove that \( \hat{x} \) is an interior point of \( Q(U) \), as desired. Indeed, suppose \( \hat{y} \in B_{X/V}(\hat{x},r) \). Picking an \( y \in X \) such that \( Qy = \hat{y} \), this means that
\[
\inf_{v \in V} \|x - y - v\| = \|\hat{x} - \hat{y}\| < r.
\]
Thus, there is some \( v \in V \) with \( \|x - y - v\| < r \). Thus, \( y + v \in B(x,r) \subseteq U \). Since the difference of \( y \) and \( y + v \) lies in \( V \), we conclude that
\[
\hat{y} = Qy = Q(y + v) \in Q(U).
\]
This proves the claim and the result follows.

**Proof of Proposition 1.** If \( T \) is continuous then, since \( Q \) is also continuous, the composition \( T \circ Q \) is also continuous. This proves the direct implication. For the other implication, assume that \( T \circ Q \) is continuous and let \( U \subseteq Y \) be open. Then \( Q^{-1}(T^{-1}(U)) = (T \circ Q)^{-1}(U) \) is also open. But then, by surjectivity of \( Q \) we have \( T^{-1}(U) = Q(Q^{-1}(T^{-1}(U))) \), which, by the lemma, is again open. This proves that \( T \) is continuous, as asserted.
By the proposition it suffices to prove that \( T \circ Q \) is continuous, where \( Q : C([0,1]) \to C([0,1])/J \) is the quotient map. Since \( (T \circ Q)(f) = f|_{[a,b]} \), we have
\[
\|(T \circ Q)(f)\|_{\infty} = \|f|_{[a,b]}\|_{\infty} \leq \|f\|_{\infty},
\]
proving the result.

Finally we need to show that \( T \) has a continuous inverse. The inverse of \( T \) is given by \( S : C([a,b]) \to C([0,1])/J, Sg := \hat{f} \), where \( f \) is defined as in (1). Note that per construction, \( f \) and \( g \) have the same range. In particular, they have the same supremum. Hence,
\[
\|Sg\| = \|\hat{f}\| \leq \|f\|_{\infty} = \|g\|_{\infty},
\]
proving that \( S \) is continuous. The result follows.

**Exercise 3: A quotient of \( C^1([0,1]) \).**
Let \( C^1([0,1]) \) be defined as in Exercise 3 of Exercise sheet 3 and fix \( p \in [0,1] \).

(a) Show that \( J := \{f \in C^1[0,1]: f(p) = f'(p) = 0\} \) is a closed subset of \( C^1([0,1]) \).

Let \( Q : C^1([0,1]) \to C^1([0,1])/J \) be the quotient map. We define a product operation on \( C^1([0,1])/J \) by \( Q(f) \cdot Q(g) := Q(fg) \) for \( f, g \in C^1([0,1]) \).

(b) Prove that this product operation is well-defined, i.e., show that if \( Q(f) = Q(\tilde{f}) \) and \( Q(g) = Q(\tilde{g}) \), then \( Q(f) \cdot Q(g) = Q(\tilde{f}) \cdot Q(\tilde{g}) \).

Let \( V \) denote the 2-dimensional subspace of the space \( \mathbb{R}^{2 \times 2} \) of real \( 2 \times 2 \) matrices spanned by the matrices \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). Define the map
\[
\phi : C^1([0,1])/J \to V, \quad \phi(Q(f)) := \begin{pmatrix} f(p) & f'(p) \\ 0 & f(p) \end{pmatrix}.
\]

(c) Prove that \( \phi \) is well-defined linear isomorphism that satisfies \( \phi(Q(f) \cdot Q(g)) = \phi(Q(f)) \phi(Q(g)) \), where the multiplication on the right should be interpreted as matrix multiplication.

**Solution:**

(a) Let \( (f_n)_{n \in \mathbb{N}} \) be a sequence in \( J \) that converges in \( C^1([0,1]) \) to some limit \( f \in C^1([0,1]) \). Since \( \|f - f_n\|_{\infty}, \|f' - f'_n\|_{\infty} \leq \|f - f_n\|_{C^1} \to 0 \), we find that both \( f_n \to f \) and \( f'_n \to f' \) uniformly. Since uniform convergence implies pointwise convergence, we have
\[
f(p) = \lim_{n \to \infty} f_n(p) = 0, \quad f'(p) = \lim_{n \to \infty} f'_n(p) = 0.
\]

Thus, \( f \in J \). We conclude that \( J \) is closed in \( C^1([0,1]) \).

(b) If \( Q(f) = Q(\tilde{f}) \) and \( Q(g) = Q(\tilde{g}) \), then \( f - \tilde{f}, g - \tilde{g} \in J \). Thus, \( f(p) = \tilde{f}(p), f'(p) = \tilde{f}'(p) \), and similarly for \( g \). But then
\[
(fg)(p) = f(p)g(p) = \tilde{f}(p)\tilde{g}(p) = (\tilde{f}\tilde{g})(p)
\]
and, by the Leibniz rule for differentiation,
\[
(fg)'(p) = f(p)g'(p) + f'(p)g(p) = \tilde{f}(p)\tilde{g}'(p) + \tilde{f}'(p)\tilde{g}(p) = (\tilde{f}\tilde{g})'(p).
\]
Thus indeed \( Q(f) \cdot Q(g) = Q(fg) = Q(\tilde{f}\tilde{g}) = Q(\tilde{f}) \cdot Q(\tilde{g}) \), as asserted.

— Turn the page! —
To show that $\phi$ is well-defined we need to check that if $Q(f) = Q(g)$, then $\phi(Q(f)) = \phi(Q(g))$.

If $Q(f) = Q(g)$, then $f(p) = g(p)$ and $f'(p) = g'(p)$. Hence,

\[
\phi(Q(f)) = \begin{pmatrix} f(p) & f'(p) \\ 0 & f(p) \end{pmatrix} = \begin{pmatrix} g(p) & g'(p) \\ 0 & g(p) \end{pmatrix} = \phi(Q(g)),
\]

as desired.

It is straightforward to check that $\phi$ is linear and we omit this here. We check that $\phi$ is bijective.

To see that $\phi$ is injective, suppose $\phi(Q(f)) = 0$. Then $\begin{pmatrix} f(p) & f'(p) \\ 0 & f(p) \end{pmatrix} = 0$ so that $f(p) = f'(p) = 0$, i.e., $f = f - 0 \in J$. Thus, $Q(f) = Q(0) = 0$, and injectivity of $\phi$ follows.

For surjectivity of $\phi$, let $\begin{pmatrix} a \\ 0 \end{pmatrix} \in V$. Define $f : [0, 1] \to \mathbb{R}$ by $f(x) := a + (x - p)b$. Then $f \in C^1([0, 1])$ with $f(p) = a$ and $f'(p) = b$. Hence, $\phi(Q(f)) = \begin{pmatrix} a \\ 0 \end{pmatrix}$. We conclude that $\phi$ is surjective. Thus, $\phi$ is a linear isomorphism, as desired.

Finally, we check that $\phi$ preserves the product operations. Note that for $f, g \in C^1([0, 1])$ it follows from the Leibniz rule for differentiation that

\[
\begin{pmatrix} f(p) & f'(p) \\ 0 & f(p) \end{pmatrix} \begin{pmatrix} g(p) & g'(p) \\ 0 & g(p) \end{pmatrix} = \begin{pmatrix} f(p)g(p) & f(p)g'(p) + f'(p)g(p) \\ 0 & f(p)g(p) \end{pmatrix} = \begin{pmatrix} (fg)(p) & (fg)'(p) \\ 0 & (fg)(p) \end{pmatrix}.
\]

Thus indeed $\phi(Q(f))\phi(Q(g)) = \phi(Q(f \cdot Q(g))$ and the result follows.

Exercise 4: Inclusions of Lebesgue spaces.

Let $p, q \in [1, \infty]$ with $p > q$.

(a) Show that $\ell^q \subseteq \ell^p$, with a strict inclusion.

(Hint: First prove this for $\|x\|_q = 1$.)

(b) Show that $L^p([0, 1]) \subseteq L^q([0, 1])$, with a strict inclusion.

(c) Show that both $L^p(\mathbb{R}) \nsubseteq L^q(\mathbb{R})$ and $L^q(\mathbb{R}) \nsubseteq L^p(\mathbb{R})$.

Solution: For $p \in [1, \infty]$ we define $p' \in [1, \infty]$ through the relation $\frac{1}{p} + \frac{1}{p'} = 1$.

(a) Let $x \in \ell^q$ with $\|x\|_q = 1$. Note that then for all $j \in \mathbb{N}$ we have $|x_j| \leq \|x\|_q = 1$. Thus, since $p - q > 0$, we have $|x_j|^p = |x_j|^q |x_j|^{p-q} \leq |x_j|^q$ and hence

\[
\sum_{j=1}^{N} |x_j|^p \leq \sum_{j=1}^{N} |x_j|^q \leq \|x\|_q^q = 1.
\]

Taking a sup over $N \in \mathbb{N}$ we conclude that $x \in \ell^p$ with $\|x\|_p \leq 1$.

Now suppose $x \in \ell^q$ is arbitrary. If $x = 0$ then clearly $x \in \ell^p$. Otherwise, note that $x = \frac{x}{\|x\|_q} \cdot \frac{\|x\|_q}{x}$. Since $\frac{x}{\|x\|_q}$ has $\|\cdot\|_q$-norm 1, we conclude from our previous estimate that $\|\frac{x}{\|x\|_q}\|_p \leq 1$. Hence,

\[
\|x\|_p = \|x\|_q \cdot \frac{x}{\|x\|_q} \leq \|x\|_q.
\]
Thus, we have a continuous inclusion $\ell^q \hookrightarrow \ell^p$.

To show that the inclusion is proper, we use the fact that $\sum_{j=1}^{\infty} \frac{1}{j^r}$ is finite for $s \in \mathbb{R}$ if and only if $s > 1$.

Choose $s \in \mathbb{R}$ such that $\frac{1}{p} < s < \frac{1}{q}$. Then $sp > 1$, while $sq < 1$. This implies that the sequence $(\frac{1}{j^r})_{j \in \mathbb{N}}$ lies in $\ell^p$, but not in $\ell^q$. Thus, the inclusion $\ell^q \subseteq \ell^p$ is proper. The result follows.

(b) Let $f \in L^p([0, 1])$. We abuse notation here and also write $f$ for a representative of this class. Note that $\frac{p}{q} > 1$ so that by Hölder’s inequality we have

$$\int_{[0, 1]} |f(x)|^q \, dx = \frac{\|f^q \cdot 1\|_1}{\|f\|_{\ell^p}^q} \leq \frac{\|f\|_{\ell^p}}{\|f\|_{\ell^p}^q} \left( \int_{[0, 1]} 1 \, dx \right)^{\frac{q}{p}}$$

$$= \|f\|_{\ell^p}^q \left( \int_{[0, 1]} 1 \, dx \right)^{\frac{q}{p}}$$

Hence, $f \in L^q([0, 1])$ with $\|f\|_q \leq \|f\|_p$.

To see that the inclusion is strict, one could use the fact that $\int_0^1 x^s \, dx$ is finite if and only if $s > -1$ and a similar argument as in part (a).

(c) To show that $L^q(\mathbb{R}) \nsubseteq L^p(\mathbb{R})$, we use part (b) to pick an $f \in L^q([0, 1])$ that is not in $L^p([0, 1])$. Note that then extending this function to $\mathbb{R}$ by $f(x) = 0$ if $x \notin [0, 1]$, we have found a function in $L^q(\mathbb{R})$ that is not in $L^p(\mathbb{R})$.

To show that $L^p(\mathbb{R}) \nsubseteq L^q(\mathbb{R})$, we use part (a) to pick a sequence $x \in \ell^p$ with $x \notin \ell^q$. Then consider the function $f$ given by $f(x) = 0$ when $x \in (-\infty, 1)$, and $f(x) = x_j$ whenever $x \in [j, j+1)$, i.e., $f = \sum_{j=1}^{\infty} x_j \chi_{[j,j+1)}$, where $\chi_{[j,j+1)}$ denotes the indicator function of the set $[j, j+1)$. Then

$$\int_{\mathbb{R}} |f(x)|^p \, dx = \sum_{j=1}^{\infty} |x_j|^p \int_{j}^{j+1} 1 \, dx = \|x\|_p^p < \infty.$$

so that $f \in L^p(\mathbb{R})$. Similarly we have $\int_{\mathbb{R}} |f(x)|^q \, dx = \|x\|_q^q = \infty$ and thus $f \notin L^q(\mathbb{R})$. The assertion follows.

Exercise 5: Uniform convexity of Lebesgue spaces.

Let $(X, \|\cdot\|)$ be a normed vector space. We say that $X$ is uniformly convex if for every $0 < \varepsilon < 2$ there is a $\delta > 0$ such that for all $x, y \in X$ with $\|x\| = \|y\| = 1$ we have the implication

$$\|x + y\| > 2 - \delta \quad \Rightarrow \quad \|x - y\| < \varepsilon.$$

(a) Show that if there is a $p \in (1, \infty)$ such that

$$\left\| \frac{1}{2}(x + y) \right\|^p + \frac{1}{2} \left\| (x - y) \right\|^p \leq \frac{\|x\|^p}{2} + \frac{\|y\|^p}{2},$$

then $X$ is uniformly convex.

(b) Let $(S, \mathcal{A}, \mu)$ be a measure space and let $p \in [2, \infty)$. Show that $L^p(S)$ is uniformly convex.
(Hint: First show that the inequality in (a) holds in \(\mathbb{R}\) for any \(p \in [2, \infty)\). For this you could show and use the inequalities
\[
(|a|^p + |b|^p)^{\frac{1}{p}} \leq (|a|^2 + |b|^2)^{\frac{1}{2}} \leq 2^{\frac{1}{2} - \frac{1}{p}}(|a|^p + |b|^p)^{\frac{1}{p}},
\]
valid for all \(a, b \in \mathbb{R}\).)

(c) Show that \(\ell^\infty\) is not uniformly convex.

Solution:

(a) Let \(\varepsilon \in (0, 2)\) and set \(\delta := 2 + (2^p - \varepsilon^p)^{\frac{1}{p}} > 0\). Then if \(x, y \in X\) with \(\|x\| = \|y\| = 1\) satisfy \(\|x + y\| > 2 - \delta\), we have
\[
1 = \frac{\|x\|^p}{2} + \frac{\|y\|^p}{2} \geq \frac{1}{2}(x + y)^p + \frac{1}{2}(x - y)^p > \frac{1}{2}(x - y)^p + \left(1 - \frac{\delta}{2}\right)^p.
\]
so that
\[
2^{-p}\|x - y\|^p < 1 - \left(1 - \frac{\delta}{2}\right)^p = 1 - 2^{-p}(2^p - \varepsilon^p) = 2^{-p}\varepsilon^p.
\]
Hence, \(\|x - y\| < \varepsilon\), as desired.

(b) First we prove the inequalities in the hint. Note that the first inequality is a special case of the inequality we proved in Exercise 4(a). The second inequality is a bit trickier, since a homogeneity argument yields the bound with constant \(2^{\frac{1}{2}}\) rather than with the constant \(2^{\frac{1}{2} - \frac{1}{p}}\). Instead, we prove this result using calculus.

Consider the function \(\phi : [0, \infty) \to [0, \infty), t \mapsto t^\frac{p}{2}\). Then
\[
\phi''(t) = \frac{P}{2}(\frac{P}{2} - 1)t^{\frac{P}{2} - 2} \geq 0
\]
for \(t > 0\), since \(\frac{P}{2} - 1 > 0\). Hence, the function \(\phi\) is convex and thus
\[
\left(\frac{|a|^2}{2} + \frac{|b|^2}{2}\right)^{\frac{p}{2}} = \phi(\frac{1}{2}|a|^2 + \frac{1}{2}|b|^2) \leq \frac{1}{2}\phi(|a|^2) + \frac{1}{2}\phi(|b|^2) = \frac{|a|^p}{2} + \frac{|b|^p}{2}.
\]
Raising this inequality to the power \(\frac{1}{p}\), the assertion follows.

Now, let \(x, y \in \mathbb{R}\). By subsequently applying the first inequality from the hint with \(a = \frac{x + y}{2}\), \(b = \frac{x - y}{2}\), evaluating the squares, and using (2), we obtain
\[
\left(\frac{1}{2}(x + y)^p + \frac{1}{2}(x - y)^p\right)^{\frac{p}{2}} \leq \left(\frac{1}{2}(x + y)^2 + \frac{1}{2}(x - y)^2\right)^{\frac{p}{2}} = \left(\frac{x^2}{2} + \frac{y^2}{2}\right)^{\frac{p}{2}} \leq \frac{|x|^p}{2} + \frac{|y|^p}{2},
\]
proving the desired inequality.

Let \(f, g \in L^p(S)\). Then by applying our scalar equality with \(x = f(s), y = g(s)\), we obtain
\[
\|\frac{1}{2}(f + g)\|_p^p + \|\frac{1}{2}(f - g)\|_p^p = \int_S \left(\frac{1}{2}|f(s) + g(s)|^p + \frac{1}{2}|f(s) - g(s)|^p\right) \, d\mu(s)
\]
\[
\leq \int_S \left(\frac{|f(s)|^p}{2} + \frac{|g(s)|^p}{2}\right) \, d\mu(s)
\]
\[
= \frac{\|f\|_p^p}{2} + \frac{\|g\|_p^p}{2}.
\]
Thus, by part (a), we conclude that $L^p(S)$ is uniformly convex.

(c) Set $\varepsilon = 1$ and let $\delta > 0$. Then choose $x = (1, 1, 0, 0, 0, \ldots)$, $y = (1, 0, 0, 0, \ldots)$. Note that $\|x\|_\infty = \|y\|_\infty = 1$ and moreover, $\|x + y\|_\infty = 2 > 2 - \delta$. However, $\|x - y\|_\infty = 1 \geq \varepsilon$. Thus, $\ell^\infty$ is not uniformly convex.