Remark. The exercises marked with a * can be handed in for correction in the “Funktionalanalysis” box in the atrium of building 20.30 at the latest at 14:00 on the day of the exercise class next week.

* Exercise 1: Properties of compact operators.

(a) Let $X$ and $Y$ be normed spaces and let $T \in K(X,Y)$. Show that $R(T)$ is separable.

(b) Let $X$, $Y$, and $Z$ be Banach spaces. Let $T \in K(X,Y)$ and $S \in L(Y,Z)$, where $S$ is injective. Prove that for every $\varepsilon > 0$ there is a constant $C \geq 0$ such that

$$\|Tx\| \leq \varepsilon \|x\| + C\|STx\|$$

for all $x \in X$.

Exercise 2: Examples of compact operators.

(a) Let $k$ be a continuous function $k : [0,1] \times [0,1] \to \mathbb{R}$. We define $T : C([0,1]) \to C([0,1])$ by

$$Tf(t) := \int_0^1 k(t,s) f(s) \, ds.$$ 

Show that $T$ is a compact operator.

(b) Show that the inclusion mapping $\iota : C^1([0,1]) \to C([0,1])$, $\iota(f) := f$ is a compact operator.

(c) Let $m \in C([0,1])$ with $m(x) \neq 0$ for all $x \in [0,1]$. Show that the operator $T : C([0,1]) \to C([0,1])$ given by $Tf := mf$ is not compact.

(d) Let $X := \{f \in C([0,1]) : f(0) = 0\}$. Then $(X, \| \cdot \|_\infty)$ is a Banach space. For each $t \geq 0$ we define

$$T_t f(x) = \begin{cases} f(x-t) & \text{if } x-t \in [0,1]; \\ 0 & \text{otherwise.} \end{cases}$$

Show that

(i) $T_t \in L(X)$ with $\|T_t\| \leq 1$ for all $t \geq 0$,

(ii) $T_s T_t = T_{s+t}$ for all $s,t \geq 0$,

(iii) $T_t$ is compact if and only if $t \geq 1$. 

— Turn the page! —
Exercise 3: Operators on $\ell^p$ given by infinite matrices.

Let $p \in (1, \infty)$ and define $p' \in (1, \infty)$ through the relation $\frac{1}{p} + \frac{1}{p'} = 1$. Suppose for each $j, k \in \mathbb{N}$ we are given an $a_{j,k} \in \mathbb{K}$ with

$$\gamma := \left( \sum_{j=1}^{\infty} \left( \sum_{k=1}^{\infty} |a_{j,k}|^{p'} \right)^{\frac{p}{p'}} \right)^{\frac{1}{p}} < \infty.$$  

(a) Let $x \in \ell^p$. Show that the sequence $y = (y_j)_{j \in \mathbb{N}}$ given by $y_j := \sum_{k=1}^{\infty} a_{j,k} x_k$ is again in $\ell^p$.

(b) Prove that the operator $T : \ell^p \to \ell^p$, $Tx := y$ with $y$ defined as in part (a) is bounded with $\|T\| \leq \gamma$.

(c) Show that $T$ is a compact operator.

Exercise 4: Isolated points in metric spaces.

Let $(X, d)$ be a non-empty complete metric space. A point $x \in X$ is called an isolated point if \{x\} is an open set in $X$.

(a) Show that if $X$ has no isolated points, then $X$ is uncountable.

(b) Show that if $X$ is countable, then the set of all isolated points in $X$ is dense in $X$.

*Hint: use the Baire Category Theorem.*

(Bonus) Exercise 5: Proper inclusions in $\ell^p$ spaces.

Let $p \in (1, \infty)$.

(a) Show that for any $1 < q < p$, the set $\ell^q$ is a Baire first category subset of $\ell^p$.

(b) Show that $\bigcup_{q \in (1,p)} \ell^q \neq \ell^p$. 

http://www.math.kit.edu/iana3/edu/la2019w/de