Exercise 1: Identifications of annihilators.
Let $X$ be a normed space and let $V$ be a subspace of $X$. Show that

(i) $(X/V)' \cong V^\perp$;

(ii) $V' \cong X'/V^\perp$.

Solution:

(i) Define $T : (X/V)' \to V^\perp$ by $T\phi(x) := \phi(\hat{x})$. Here $\hat{x}$ denotes the class in $X/V$ represented by $x \in X$. Note that $T$ is linear. Moreover, $T\phi$ is a linear functional on $X$, as it is the composition of the (linear) quotient map $X \to X/V$ and the linear map $\phi : X/V \to K$.

We first check that we indeed have $T\phi \in V^\perp$ whenever $\phi \in (X/V)'$. Let $v \in V$. Then $\hat{v} = 0$ in $X/V$ and hence $T\phi(v) = \phi(\hat{v}) = \phi(0) = 0$, proving that $T\phi$ indeed vanishes on $V$. Next, we check that it is bounded. Note that for all $x \in X$ we have

$$|T\phi(x)| = |\phi(\hat{x})| \leq \|\phi\| \|\hat{x}\| \leq \|\phi\| \|x\|$$

so that $T\phi$ is bounded with

$$\|T\phi\| \leq \|\phi\|. \quad (1)$$

Thus, $T\phi \in V^\perp$, and $T$ is well-defined.

Next, we provide an inverse for $T$. Define $S : V^\perp \to (X/V)'$ by $S\psi(\hat{x}) := \psi(x)$. To see that this is well-defined, note that if $x, y \in X$ satisfy $\hat{x} = \hat{y}$, i.e., $x - y \in V$, then $\psi(x) - \psi(y) = \psi(x - y) = 0$, since $\psi \in V^\perp$, as desired. Moreover, note that for any $v \in V$ we have

$$|S\psi(\hat{x})| = |\psi(x)| = |\psi(x - v)| \leq \|\psi\| \|x - v\|.$$ 

Taking an infimum over $v \in V$ proves that $|S\psi(\hat{x})| \leq \|\psi\| \|\hat{x}\|$, proving that indeed $S\psi \in (X/V)'$, and moreover that

$$\|S\psi\| \leq \|\psi\|. \quad (2)$$

To see that $S$ is an inverse for $T$, note that for any $\phi \in (X/V)'$ and $x \in X$ we have $S(T\phi)(\hat{x}) = T\phi(x) = \phi(\hat{x})$ so that $ST = \text{Id}$. Moreover, we have $T(S\psi)(x) = S\psi(\hat{x}) = \psi(x)$ so that $TS = \text{Id}$. We conclude that $S = T^{-1}$ and $T$ is a linear isomorphism. It remains to show that $T$ is an isometry.

Let $\phi \in (X/V)'$. From (2) it follows that $\|\phi\| = \|ST\phi\| \leq \|T\phi\|$. Combining this with (1) proves that $\|T\phi\| = \|\phi\|$, as desired. The assertion follows.
(ii) We define $T : V' \to X'/V^\perp$ as follows. Let $\phi \in V'$. Then by Hahn-Banach, there exists a $\psi \in X'$ such that $\psi(v) = \phi(v)$ for all $v \in V$ and $\|\psi\| = \|\phi\|$. Now set $T\phi := \hat{\psi}$, where $\hat{\psi}$ denotes the class in $X'/V^\perp$ represented by $\psi$. To see that this is well-defined, we need to show that $T\phi$ does not depend on the particular extension chosen for $\phi$. Indeed, suppose that $\psi_1, \psi_2 \in X'$ are both extensions of $\phi$ to $X$. Then for each $v \in V$ we have

$$ (\psi_1 - \psi_2)(v) = \psi_1(v) - \psi_2(v) = \phi(v) - \phi(v) = 0. $$

Hence, $\psi_1 - \psi_2 \in V^\perp$ and thus $\hat{\psi}_1 = \hat{\psi}_2$, as desired. Finally, note that we have

$$ \|T\phi\| = \|\hat{\psi}\| \leq \|\psi\| = \|\phi\|. \quad (3) $$

Next, we define $S : X'/V^\perp \to V'$ by $S\hat{\psi}(v) := \psi(v)$. To see that this is well defined, note that if $\psi_1 - \psi_2 \in V^\perp$, then for all $v \in V$ we have $\psi_1(v) - \psi_2(v) = (\psi_1 - \psi_2)(v) = 0$ so that $\psi_1(v) = \psi_2(v)$, as desired. Next, note that for any $\hat{\psi} \in V^\perp$ we have

$$ \|S\hat{\psi}\| = \|S(\hat{\psi} - \hat{\psi})\| = \sup_{v \in V \atop \|v\|=1} \|\psi(v) - \hat{\psi}(v)\| \leq \sup_{x \in X \atop \|x\|=1} \|\psi(x) - \hat{\psi}(x)\| = \|\psi - \hat{\psi}\|. $$

Taking an infimum over $\hat{\psi} \in V^\perp$ we conclude that

$$ \|S\hat{\psi}\| \leq \|\hat{\psi}\|. \quad (4) $$

Next we check that $S$ is the inverse of $T$. Indeed, note that for any $\phi \in V'$ with extensions $\psi \in X'$, we have $S(T\phi)(v) = S\hat{\psi}(v) = \psi(v) = \phi(v)$ for all $v \in V$. Hence, $ST = \text{Id}$.

Furthermore, note that for $\psi \in X'$, we have that $\psi$ is an extension of $S\hat{\psi} = \psi|_V$ to $X$. Thus, $T(S\hat{\psi}) = \hat{\psi}$, proving that $TS = \text{Id}$ and thus, that $T$ is a linear isomorphism. Finally, as in (i), we deduce from (3) and (4) that $T$ is an isometry. The assertion follows.

**Exercise 2: Duality in sequence spaces.**

For $j \in \mathbb{N}$ we denote by $e_j$ the sequence which is 0 in every entry except for the $j$-th one, where it is equal to 1.

(a) Prove the following assertions:

(i) For each $x \in c_0$, we have $\sum_{j=1}^J x_j e_j \to x$ in $c_0$ as $J \to \infty$.

(ii) Let $p \in [1, \infty)$. For each $x \in \ell^p$, we have $\sum_{j=1}^J x_j e_j \to x$ in $\ell^p$ as $J \to \infty$.

(iii) There exists an $x \in \ell^\infty$ so that $\sum_{j=1}^J x_j e_j$ does not converge to $x$ in $\ell^\infty$ as $J \to \infty$.

(b) Prove that we have the following identifications:

(i) $(c_0)' \cong \ell^1$;

(ii) for $p \in [1, \infty)$, $(\ell^p)' \cong \ell^{p'}$.

*Hint: use Exercise 3 of Exercise sheet 7."

(c) Show that there exists a functional $\phi \in (\ell^\infty)'$ such that for every convergent sequence $x = (x_j)_{j \in \mathbb{N}}$ we have $\phi(x) = \lim_{j \to \infty} x_j$. \(\textbf{Bonus:}\) prove or disprove that such a functional is unique.)
Furthermore, show that there is no \( y \in \ell^1 \) such that \( \phi(x) = \sum_{j=1}^{\infty} x_j y_j \) for all \( x \in \ell^\infty \).

**Solution:**

(a) For (i), let \( x \in c_0 \). Then

\[
\|x - \sum_{j=1}^{J} x_j e_j\|_\infty = \|(0, \ldots, 0, x_{J+1}, x_{J+2}, \ldots)\|_\infty = \sup_{j > J} |x_j| \to \sup_{j \to \infty} |x_j| = \lim_{j \to \infty} |x_j| = 0
\]
as \( J \to \infty \), as desired.

For (ii), let \( x \in \ell^p \). Then

\[
\|x - \sum_{j=1}^{J} x_j e_j\|_p = \left( \sum_{j=1}^{J} |x_j|^p \right)^{\frac{1}{p}} \to 0 \quad \text{as} \quad J \to \infty,
\]
as desired.

For (iii), we could take any \( x \in \ell^\infty \setminus c_0 \), since the solution to (i) shows that only sequences that converge to 0 satisfy the required convergence property. To give a concrete example, we let \( x := (1, 1, \ldots) \in \ell^\infty \). Then

\[
\|x - \sum_{j=1}^{J} x_j e_j\|_\infty = 1 \quad \text{for all} \quad J \in \mathbb{N}
\]
and thus \( \sum_{j=1}^{J} x_j e_j \) does not converge to \( x \).

(b) (i) We define \( T : \ell^1 \to (c_0)' \) as follows. For \( y \in \ell^1 \) we define

\[
Ty : c_0 \to \mathbb{K}, \quad Ty(x) := \sum_{j=1}^{\infty} x_j y_j.
\]

Then \( Ty \) is linear, and by Hölder’s inequality we have \( |Ty(x)| \leq \|x\| \|y\|_1 \) so that \( Ty \in (c_0)' \) with

\[
\|Ty\| \leq \|y\|_1. \tag{5}
\]

Thus, \( T \) is well-defined.

Next, we will find an inverse of \( T \). Define \( S : (c_0)' \to \ell^1 \) by \( S \phi := (\phi(e_j))_{j \in \mathbb{N}} \). We use Exercise 3 of Exercise sheet 7 to see that \( S \) is well-defined. We define \( y_j := \phi(e_j) \), and we need to check that \( y \in \ell^1 \). To this end let \( x \in c_0 \). Then by the first part of (a) we have \( \sum_{j=1}^{J} x_j e_j \to x \) in \( c_0 \) as \( J \to \infty \). Since \( \phi : c_0 \to \mathbb{K} \) is continuous, this implies that

\[
\lim_{J \to \infty} \sum_{j=1}^{J} x_j y_j = \lim_{J \to \infty} \sum_{j=1}^{J} x_j \phi(e_j) = \lim_{J \to \infty} \phi \left( \sum_{j=1}^{J} x_j e_j \right) = \phi(x). \tag{6}
\]

Thus, the series \( \sum_{j=1}^{\infty} x_j y_j \) converges for any \( x \in c_0 \). By part (a) of Exercise 3 of Exercise sheet 7 this implies that \( y \in \ell^1 \), as desired. Moreover, in (the solution to) this exercise it was shown that

\[
\|y\|_1 \leq \sup_{n \in \mathbb{N}} \sup_{x \in c_0, \|x\|_\infty = 1} \left| \sum_{j=1}^{n} x_j y_j \right| = \sup_{n \in \mathbb{N}} \sup_{x \in c_0, \|x\|_\infty = 1} |\phi\left( \sum_{j=1}^{n} x_j e_j \right)|
\]

\[
\leq \|\phi\| \sup_{n \in \mathbb{N}} \sup_{x \in c_0, \|x\|_\infty = 1} \max_{j \in \{1, \ldots, n\}} |x_j| = \|\phi\| \sup_{x \in c_0, \|x\|_\infty = 1} \|x\| = \|\phi\|.
\]
Hence,
\[ \| S \phi \|_1 = \| y \|_1 \leq \| \phi \|. \]  
(7)

Finally, note that \( S(Ty) = (Ty(e_j))_{j \in \mathbb{N}} = (y_j)_{j \in \mathbb{N}} = y \) so that \( ST = \text{Id} \). Moreover, it follows from (6) that
\[ T(S\phi)(x) = \sum_{j=1}^{\infty} x_j \phi(e_j) = \phi(x) \]
for all \( x \in c_0 \) so that \( T(S\phi) = \phi \) and thus \( TS = \text{Id} \). We conclude that \( S = T^{-1} \) and \( T \) is a linear isomorphism. Moreover, it follows from (5) and (7) that \( T \) is an isometry. The assertion follows.

(ii) This result is completely analogous to the proof of (i), this time using the second part of (a), and part (b) of Exercise 3 of Exercise sheet 7.

(c) Denote by \( c \subseteq \ell^\infty \) the space of convergent sequences and define \( \psi : c \to \mathbb{K} \) by \( \psi(x) = \lim_{j \to \infty} x_j \). By the usual properties of limits, \( \psi \) is linear. Moreover, we have
\[ |\psi(x)| = \lim_{j \to \infty} |x_j| \leq \sup_{j \in \mathbb{N}} |x_j| = \| x \|_\infty. \]
Thus, \( \psi \) is bounded and hence \( \phi \in c' \). Thus, by Hahn-Banach this map extends to a functional \( \phi \in (\ell^\infty)' \). Then for any convergent sequence \( x \) we have \( \phi(x) = \lim_{j \to \infty} x_j \) and thus, \( \phi \) satisfies the required property.

Suppose for a contradiction that there is a sequence \( y \in \ell^1 \) such that \( \phi(x) = \sum_{j=1}^{\infty} x_j y_j \) for all \( x \in \ell^\infty \). Note that for a fixed \( j \in \mathbb{N} \) the sequence \( e_j \) converges to 0. Hence,
\[ y_j = \phi(e_j) = 0 \quad \text{for all } j \in \mathbb{N} \]
and thus \( y = 0 \) so that \( \phi = 0 \). Taking the sequence \( x = (1, 1, \ldots) \), we then find that \( 0 = \phi(x) = 1 \), which is a contradiction. The assertion follows.

For the bonus problem, we will show that such a \( \phi \) is not unique. Consider the spaces \( c_+ \subseteq \ell^\infty \) of sequences \( (x_j)_{j \in \mathbb{N}} \) with the property that the subsequence \( (x_{2j})_{j \in \mathbb{N}} \) converges and the space \( c_- \subseteq \ell^\infty \) of sequences \( (x_j)_{j \in \mathbb{N}} \) with the property that the subsequence \( (x_{2j+1})_{j \in \mathbb{N}} \) converges. Respectively defining \( \psi_+ : c_+ \to \mathbb{K} \), \( \psi_-(x) := \lim_{j \to \infty} x_{2j} \), \( \psi_-(x) := \lim_{j \to \infty} x_{2j+1} \), we note as before that \( \| \psi_\pm (x) \| \leq \| x \|_\infty \) and thus, by Hahn-Banach these map have bounded extensions \( \phi_\pm \in (\ell^\infty)' \). Then certainly \( \phi_+ (x) = \lim_{j \to \infty} x_j \) whenever \( x = (x_j)_{j \in \mathbb{N}} \) is convergent, but for the sequence \( ((-1)^j)_{j \in \mathbb{N}} \in \ell^\infty \) we have \( \phi_+(x) = 1 \), while \( \phi_-(x) = -1 \). Hence, \( \phi_+ \neq \phi_- \), and thus, a map with the property we are looking for is not uniquely determined.

Exercise 3: Projections and products.
Let \( X \) be a Banach space.

(a) Let \( P : X \to X \) be a linear map satisfying \( P^2 = P \). Show that the following are equivalent:

(i) \( P \) is bounded;

(ii) \( N(P) \) and \( R(P) \) are closed.

(Hint: for (ii)\( \Rightarrow \)(i), use the Closed Graph Theorem.)
(b) Suppose $X$ decomposes as $X = X_1 \oplus X_2$ for closed subspaces $X_1, X_2 \subseteq X$. Show that there exists a continuous linear isomorphism between $X$ and $X_1 \times X_2$ with a continuous inverse. Here we equip the cartesian product $X_1 \times X_2$ with the norm $\|(x_1, x_2)\| := \|x_1\| + \|x_2\|$. 

Solution:

(a) For (i)⇒(ii), note that $N(P) = P^{-1}(\{0\})$ is closed, since $P$ is continuous. Moreover, since $P^2 = P$, we find that $R(P) = \{x \in X : Px = x\} = (\text{Id} - P)^{-1}(\{0\})$, which is also closed, since $\text{Id} - P$ is also continuous.

For (ii)⇒(i) we use the Closed Graph Theorem. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $X$ such that for $x, y \in X$ we have $x_n \to x$ and $Px_n \to y$ as $n \to \infty$. To conclude that $P$ is bounded, it now suffices to show that $Px = y$.

Since $R(P)$ is closed and $(Px_n)_{n \in \mathbb{N}}$ is a sequence in $R(P)$, its limit $y$ also lies in $R(P)$. Thus, we have $P_y = y$. Moreover, since $N(P)$ is closed and $(x_n - Px_n)_{n \in \mathbb{N}}$ lies in $N(P)$ (we have $P(x_n - Px_n) = Px_n - P^2x_n = Px_n - Px_n = 0$), its limit $x - y$ also lies in $N(P)$. Combining these facts, we conclude that, $0 = P(x - y) = Px - Py = Px - y$ and thus $Px = y$, as desired.

(b) We define $T : X_1 \times X_2 \to X$ by $T(x_1, x_2) = x_1 + x_2$. Then $T$ is surjective as $X = X_1 \oplus X_2$ and $T$ is injective as the only way to write $x_1 + x_2 = 0$ for $x_1 \in X_1, x_2 \in X_2$ is when $x_1 = 0$ and $x_2 = 0$. Moreover, $T$ is continuous, since

$$\|T(x_1, x_2)\| = \|x_1 + x_2\| \leq \|x_1\| + \|x_2\| = \|(x_1, x_2)\|$$

by the triangle inequality.

We can now conclude the result in at least two different ways. One way is to use the Open Mapping Theorem to conclude that the inverse of $T$ is also continuous, proving the desired result.

Alternatively, we could use part (a) of this exercise. By Exercise 6 of Exercise sheet 4, the maps $P_1 x = x_1$ and $P_2 x = x_2$, where $x = x_1 + x_2$ is the unique decomposition of $x$ with $x_1 \in X_1$ and $x_2 \in X_2$, are the unique projections satisfying $R(P_1) = N(P_2) = X_1$, $N(P_1) = R(P_2) = X_2$. By (a) these maps are continuous. Noting that for $x \in X$ we have $T^{-1}x = (P_1 x, P_2 x)$, we find that

$$\|T^{-1}x\| = \|(P_1 x, P_2 x)\| = \|P_1 x\| + \|P_2 x\| \leq (\|P_1\| + \|P_2\|)\|x\|,$$

proving that $T^{-1}$ is continuous. The assertion follows.

We point out that in particular we have shown that the norms $\|x_1\| + \|x_2\|$ and $\|x_1 + x_2\|$ are equivalent norms on $X$.

Exercise 4: A bounded operator.
Let $X$ be a normed space and let $(x_j)_{j \in \mathbb{N}}$ be a sequence in $X$. Let $p \in [1, \infty]$ and suppose that for each $\phi \in X'$ we have $(\phi(x_j))_{j \in \mathbb{N}} \in \ell^p$. Prove that $T : X' \to \ell^p$, $T\phi := (\phi(x_j))_{j \in \mathbb{N}}$ is a bounded operator.

(Hint: use the Closed Graph Theorem.)
Solution: Since $X'$ and $\ell^p$ are Banach spaces we may use the Closed Graph Theorem to prove that $T$ is bounded. Suppose $(\phi_n)_{n \in \mathbb{N}}$ is a sequence in $X'$ such that for $\phi \in X'$, $y \in \ell^p$ we have $\phi_n \to \phi$ in $X'$ and $T\phi_n \to y$ in $\ell^p$ as $n \to \infty$. It remains to show that $T\phi = y$.

Fix a $j \in \mathbb{N}$. Then we have

$$|\phi_n(x_j) - y_j| \leq \|T\phi_n - y\|_\infty \leq \|T\phi_n - y\|_p \to 0 \quad \text{as } n \to \infty$$

so that $\lim_{n \to \infty} \phi_n(x_j) = y_j$ in $\mathbb{K}$. But since $\phi_n \to \phi$ in $X'$, we also have

$$|\phi_n(x_j) - \phi(x_j)| = |(\phi_n - \phi)(x_j)| \leq \|\phi_n - \phi\||x_j| \to 0 \quad \text{as } n \to \infty$$

so that $\lim_{n \to \infty} \phi_n(x_j) = \phi(x_j)$ in $\mathbb{K}$. By uniqueness of limits, we conclude that $\phi(x_j) = y_j$.
But then $T\phi = (\phi(x_j))_{j \in \mathbb{N}} = (y_j)_{j \in \mathbb{N}} = y$, proving the desired result.