Functional Analysis

Exercise Sheet 9

Remark. The exercises marked with a * can be handed in for correction in the “Funktionalanalyse” box in the atrium of building 20.30 at the latest at 14:00 on the day of the exercise class next week.

Exercise 1: Separation of closed convex sets
Let $X$ be a normed space and let $K$, $L$ be disjoint closed convex subsets of $X$. Assume that $K$ is compact.

(a) Show that there exists an $r > 0$ such that $K + B(0, r) := \{x + y : x \in K, y \in B(0, r)\}$ and $L$ are disjoint.

(b) Show that there exists a $\phi \in X'$ such that
$$\sup_{x \in K} \Re \phi(x) < \inf_{x \in L} \Re \phi(x).$$

* Exercise 2: Separation of concave and convex functions
Let $U \subseteq \mathbb{R}^n$ be an open convex set. A function $f : U \to \mathbb{R}$ is called convex if for all $x, y \in U$ and $t \in [0, 1]$ we have $f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$. Moreover, $f$ is called concave if $-f$ is convex. If $f$ is both convex and concave, then $f$ is called affine.

Let $f : U \to \mathbb{R}$ be a continuous concave function and $g : U \to \mathbb{R}$ a continuous convex function satisfying $f(x) \leq g(x)$ for all $x \in U$. Show that there exists an affine function $h : U \to \mathbb{R}$ satisfying $f(x) \leq h(x) \leq g(x)$ for all $x \in U$.

(Hint: consider the sets $\{(x, r) \in U \times \mathbb{R} : r < f(x)\}$, $\{(x, r) \in U \times \mathbb{R} : r > g(x)\}$.)

Exercise 3: Inner products
Show that in the following settings, the map $(\cdot, \cdot) : X \times X \to \mathbb{K}$ is an inner products on $X$.

(i) $X = C^1([0, 1]; \mathbb{K})$, i.e., the space of $C^1$ functions $f : [0, 1] \to \mathbb{K}$, with
$$ (f, g) := \int_0^1 f'(t)\overline{g'(t)} \, dt + \int_0^1 f(t)\overline{g(t)} \, dt. $$

(ii) $X = \mathbb{K}^{n \times n}$, i.e., the space of $n \times n$ matrices with coefficients in $\mathbb{K}$, with
$$ (A, B) = \text{tr}(B^*A). $$

Here $B^*$ is the conjugate transpose of $B$ and tr denotes the trace.
(iii) \( X = L^2(\mathbb{R}; \mathbb{R}^n) \), i.e., the space of those measurable functions \( f : \mathbb{R} \to \mathbb{R}^n \) such that \( |f|^2 \) is integrable over \( \mathbb{R} \), where \( | \cdot | \) denotes the Euclidean norm on \( \mathbb{R}^n \). For a positive definite matrix \( A \in \mathbb{R}^{n \times n} \) we set
\[
(f, g) = \int_{\mathbb{R}} Af(t) \cdot g(t) \, dt.
\]

**Exercise 4: Cauchy-Schwarz inequality**

Let \( X \) be a vector space over \( \mathbb{K} \) with inner product \( (\cdot, \cdot) \) and norm \( \|x\|^2 = (x, x) \). Give a proof of the Cauchy-Schwarz inequality \( |(x, y)| \leq \|x\|\|y\| \) for all \( x, y \in X \). Moreover, show that one has equality in this inequality if and only if \( x \) and \( y \) are linearly dependant.

(*Hint: there are many different proofs. One that you could try that is “different” from the one in the book is expanding the inequality \( 0 \leq \|x - y\|^2 \). Then use homogeneity.*)

**Exercise 5: Continuity of bilinear maps**

Let \( X, Y, Z \) be normed spaces and let \( b : X \times Y \to Z \) be a bilinear mapping, i.e., a mapping such that for each fixed \( x_0 \in X, y_0 \in Y \) the mappings
\[
b(x_0, \cdot) : Y \to Z, \quad y \mapsto b(x_0, y) \\
b(\cdot, y_0) : X \to Z, \quad x \mapsto b(x, y_0)
\]
are linear.

We equip \( X \times Y \) with the norm \( \|(x, y)\| := \|x\| + \|y\| \). Suppose that \( X \) or \( Y \) is a Banach space. Show that the following are equivalent:

(i) \( b \) is continuous;

(ii) for each \( x_0 \in X, y_0 \in Y \) the mappings \( b(x_0, \cdot), b(\cdot, y_0) \) are continuous;

(iii) there is a \( c \geq 0 \) such that \( \|b(x, y)\| \leq c\|x\|\|y\| \) for all \( x \in X, y \in Y \).

(*Hint: for (ii)\(\Rightarrow\)(iii), use the Uniform Boundedness Principle.*)

**Bonus:** Show that (i) and (iii) are equivalent and imply (ii) even if we do not require \( X \) or \( Y \) to be a Banach space. Give an example of normed spaces \( X, Y, Z \) and a bilinear map \( b : X \times Y \to Z \) that satisfies (ii) but not (i) and (iii).