Functional Analysis

Solutions to exercise sheet 9

Exercise 1: Separation of closed convex sets
Let $X$ be a normed space and let $K, L$ be disjoint closed convex subsets of $X$. Assume that $K$ is compact.

(a) Show that there exists an $r > 0$ such that $K + B(0, r) := \{x + y : x \in K, y \in B(0, r)\}$ and $L$ are disjoint.

(b) Show that there exists a $\phi \in X'$ such that
$$\sup_{x \in K} \Re \phi(x) < \inf_{x \in L} \Re \phi(x).$$

Solution:

(a) One can use the fact here that in any metric space the distance between a closed and a compact set that are disjoint is positive. Then set $r := \text{dist}(K, L) > 0$. For those unfamiliar with this result, we prove this directly here.

Suppose that there is no $r > 0$ such that $K + B(0, r)$ and $L$ are disjoint. Then for each $n \in \mathbb{N}$ we can find an $x_n \in (K + B(0, \frac{1}{n})) \cap L$. Write $x_n = k_n + b_n$, with $k_n \in K$ and $b_n \in B(0, \frac{1}{n})$. Note that then $\|b_n\| < \frac{1}{n} \to 0$ such that $b_n \to 0$ in $X$ as $n \to \infty$. Moreover, since $K$ is compact, there is a $k \in K$ and a subsequence $(k_{n_j})_{j \in \mathbb{N}}$ such that $k_{n_j} \to k$ as $j \to \infty$. But then $x_{n_j} = k_{n_j} + b_{n_j} \to k + 0 = k$ as $j \to \infty$.

Since $L$ is closed and $(x_{n_j})_{j \in \mathbb{N}}$ is a sequence in $L$, we also have $k \in L$. But then $k \in L \cap K = \emptyset$, which is absurd. The assertion follows.

(b) Note that $K + B(0, r) = \bigcup_{y \in K} B(x, r)$ is an open set, as it is the union of open sets. Hence, by the Hahn-Banach Separation Theorem, there is a functional $\phi \in X'$ such that
$$\Re \phi(k + b) < \Re \phi(x)$$
for all $k \in K, b \in B(0, r)$ and $x \in L$. Thus, fixing $x \in L$ and $k \in K$ we have
$$\Re \phi(k) + r \Re \phi(y) = \Re \phi(k) + \Re \phi(ry) < \Re \phi(x)$$
for all $y \in Y$ with $\|y\| < 1$. Taking a supremum over such $y$, we find that
$$\Re \phi(k) + r\|\Re \phi\| \leq \Re \phi(x).$$
Thus, taking a supremum over $k \in K$ and an infimum over $x \in L$ we conclude that
$$\sup_{x \in K} \Re \phi(x) < \sup_{x \in K} \Re \phi(x) + r\|\Re \phi\| \leq \inf_{x \in L} \Re \phi(x).$$
Exercise 2: Separation of concave and convex functions

Let \( U \subseteq \mathbb{R}^n \) be an open convex set. A function \( f : U \to \mathbb{R} \) is called convex if for all \( x, y \in U \) and \( t \in [0, 1] \) we have \( f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \). Moreover, \( f \) is called concave if \( -f \) is convex. If \( f \) is both convex and concave, then \( f \) is called affine.

Let \( f : U \to \mathbb{R} \) be a continuous concave function and \( g : U \to \mathbb{R} \) a continuous convex function satisfying \( f(x) \leq g(x) \) for all \( x \in U \). Show that there exists an affine function \( h : U \to \mathbb{R} \) satisfying \( f(x) \leq h(x) \leq g(x) \) for all \( x \in U \).

(Hint: consider the sets \( \{(x, r) \in U \times \mathbb{R} : r < f(x)\} \cup \{(x, r) \in U \times \mathbb{R} : r > g(x)\} \).

Solution: Let \( V_1 := \{(x, r) \in U \times \mathbb{R} : r < f(x)\} \) and \( V_2 := \{(x, r) \in U \times \mathbb{R} : r > g(x)\} \). Then these sets are convex subsets of \( \mathbb{R}^{n+1} \), and \( V_1 \) is open. For this latter result, define \( \tilde{f} : U \times \mathbb{R} \to \mathbb{R} \) by \( \tilde{f}(x, r) := f(x) - r \). Then this function is continuous since \( f \) is, and \( V_1 = \tilde{f}^{-1}((0, \infty)) \), proving that this set is open in \( U \times \mathbb{R} \). Since \( U \) is open in \( \mathbb{R}^n \), we conclude that \( V_1 \) is open in \( \mathbb{R}^{n+1} \).

To see that \( V_1 \) is convex, note that if \( (x_0, r_0), (x_1, r_1) \in V_1 \), then, since \( f \) is concave, for all \( t \in [0, 1] \),
\[
    tr_1 + (1-t)r_0 < tf(x_1) + (1-t)f(x_0) \leq f(tx_1 + (1-t)x_0).
\]
Hence, \( t(x_1, r_1) + (1-t)(x_0, r_0) \in V_1 \), as desired. The proof of convexity for \( V_2 \) is similar.

Finally we note that \( V_1 \) and \( V_2 \) are disjoint. Indeed, if \( (x, r) \in V_1 \), then \( r < f(x) \leq g(x) \) and hence, \( (x, r) \notin V_2 \). Thus indeed, \( V_1 \cap V_2 = \emptyset \).

By the Hahn-Banach Separation Theorem we can find a functional \( \phi : \mathbb{R}^{n+1} \to \mathbb{R} \) such that \( \phi(x, r) < \phi(y, s) \) for all \( (x, r) \in V_1 \), \( (y, s) \in V_2 \). Pick \( \alpha \in \mathbb{R} \) such that
\[
    \phi(x, r) \leq \alpha \leq \phi(y, s) \quad \text{for all} \quad (x, r) \in V_1, \ (y, s) \in V_2, \tag{1}
\]
i.e., we pick \( \alpha \in [\sup_{(x,r)\in V_1} \phi(x,r), \inf_{(y,s)\in V_2} \phi(y,s)] \). Then we wish to define the function \( h : U \to \mathbb{R} \) as the function whose graph is given by \( G := \{(x, r) \in U \times \mathbb{R} : \phi(x, r) = \alpha \} \). To see that \( G \) is indeed the graph of a function, we need to check that if \( (x, r), (x, s) \in G \), then \( r = s \). To this end, we first note that all functionals on \( \mathbb{R}^{n+1} \) are given by taking inner products with vectors in \( \mathbb{R}^{n+1} \). Thus there are \( z \in \mathbb{R}^n \), \( w \in \mathbb{R} \) such that \( \phi(x, r) = z \cdot x + wr \) for all \( (x, r) \in \mathbb{R}^{n+1} \).

Let \( x \in U \) and \( \varepsilon > 0 \). Then \( (x, f(x) - \varepsilon) \in V_1 \) and \( (x, g(x)) \in V_2 \) so that
\[
    z \cdot x + w(f(x) - \varepsilon) = \phi(x, f(x) - \varepsilon) < \phi(x, g(x)) = z \cdot w + wg(x).
\]
Thus we have \( w(f(x) - \varepsilon) < wg(x) \) and hence \( 0 < w(g(x) - f(x) + \varepsilon) \). Since \( g(x) - f(x) + \varepsilon \geq \varepsilon > 0 \), we conclude that \( w > 0 \).

Finally, if \( (x, r), (x, s) \in G \), then \( \phi(x, r) = \alpha = \phi(x, s) \) and hence \( 0 = \phi(0, r - s) = w(r - s) \). Since \( w > 0 \), we conclude that \( r = s \), as desired. Thus \( G \) is indeed the graph of a function which we denote by \( h \). But then for all \( x \in U \) and \( \varepsilon > 0 \) we have \( z \cdot x + wh(x) = \phi(x, h(x)) = \alpha \) so that by (1) we have
\[
    z \cdot x + w(f(x) - \varepsilon) \leq z \cdot x + wh(x) \leq z \cdot x + wg(x).
\]
Hence, since \( w > 0 \), we have \( f(x) - \varepsilon \leq h(x) \leq g(x) \). Letting \( \varepsilon \to 0 \), we conclude that \( f \leq h \leq g \).
Finally, we show that $h$ is affine. Indeed, if $x, y \in U$ and $t \in [0,1]$, then
\[
\phi(tx + (1-t)y, th(x) + (1-t)h(y)) = t\phi(x, h(x)) + (1-t)\phi(y, h(y)) = t\alpha + (1-t)\alpha = \alpha
\]
so that $(tx + (1-t)y, th(x) + (1-t)h(y)) \in G$. Since also $(tx + (1-t)y, h(tx + (1-t)y)) \in G$ we conclude that $h(tx + (1-t)y) = th(x) + (1-t)h(y)$. The assertion follows.

Exercise 3: Inner products
Show that in the following settings, the map $(\cdot, \cdot) : X \times X \to \mathbb{K}$ is an inner products on $X$.

(i) $X = C^1([0,1]; \mathbb{K})$, i.e., the space of $C^1$ functions $f : [0,1] \to \mathbb{K}$, with
\[
(f, g) := \int_0^1 f'(t)g'(t)\,dt + \int_0^1 f(t)g(t)\,dt.
\]

(ii) $X = \mathbb{K}^{n \times n}$, i.e., the space of $n \times n$ matrices with coefficients in $\mathbb{K}$, with
\[
(A, B) = \text{tr}(B^*A).
\]
Here $B^*$ is the conjugate transpose of $B$ and $\text{tr}$ denotes the trace.

(iii) $X = L^2(\mathbb{R}; \mathbb{R}^n)$, i.e., the space of those measurable functions $f : \mathbb{R} \to \mathbb{R}^n$ such that $|f|^2$ is integrable over $\mathbb{R}$, where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$. For a positive definite matrix $A \in \mathbb{R}^{n \times n}$ we set
\[
(f, g) = \int_{\mathbb{R}} Af(t) \cdot g(t)\,dt.
\]

Solution: This exercise is not particularly deep and we leave it to the reader.

Exercise 4: Cauchy-Schwarz inequality
Let $X$ be a vector space over $\mathbb{K}$ with inner product $(\cdot, \cdot)$ and norm $\|x\|^2 = (x, x)$. Give a proof of the Cauchy-Schwarz inequality $|\langle x, y \rangle| \leq \|x\|\|y\|$ for all $x, y \in X$. Moreover, show that one has equality in this inequality if and only if $x$ and $y$ are linearly dependant.

(Hint: there are many different proofs. One that you could try that is “different” from the one in the book is expanding the inequality $0 \leq \|x - y\|^2$. Then use homogeneity.)

* Exercise 5: Continuity of bilinear maps
Let $X, Y, Z$ be normed spaces and let $b : X \times Y \to Z$ be a bilinear mapping, i.e., a mapping such that for each fixed $x_0 \in X$, $y_0 \in Y$ the mappings
\[
b(x_0, \cdot) : Y \to Z, \quad y \mapsto b(x_0, y) \\
b(\cdot, y_0) : X \to Z, \quad x \mapsto b(x, y_0)
\]
are linear.

We equip $X \times Y$ with the norm $\|(x, y)\| := \|x\| + \|y\|$. Suppose that $X$ or $Y$ is a Banach space. Show that the following are equivalent:

(i) $b$ is continuous:
(ii) for each \( x_0 \in X, y_0 \in Y \) the mappings \( b(x_0, \cdot), b(\cdot, y_0) \) are continuous;

(iii) there is a \( c \geq 0 \) such that \( \|b(x, y)\| \leq c\|x\|\|y\| \) for all \( x \in X, y \in Y \).

(Hint: for (ii)⇒(iii), use the Uniform Boundedness Principle.)

**Bonus:** Show that (i) and (iii) are equivalent and imply (ii) even if we do not require \( X \) or \( Y \) to be a Banach space. Give an example of normed spaces \( X, Y, Z \) and a bilinear map \( b : X \times Y \to Z \) that satisfies (ii) but not (i) and (iii).

**Solution:**

“(i)⇒(ii)”

For a fixed \( x_0 \in X \) we define \( \iota_{x_0} : Y \to X \times Y \) by \( \iota_{x_0}y := (x_0, y) \). Then \( \iota_{x_0} \) is continuous (since \( \|\iota_{x_0}y - \iota_{x_0}y'\| = \|(0, y - y')\| = \|y - y'\| \), it is even Lipschitz continuous). Since \( b \) is continuous, the composition of continuous functions \( b(x_0, \cdot) = b \circ \iota_{x_0} \) is also continuous. Proving that \( b(\cdot, y_0) \) is continuous is analogous.

“(ii)⇒(iii)”

Without loss of generality, assume that \( X \) is a Banach space. We consider the family of operators \( (b(\cdot, y))_{y \in S_Y} \) in \( L(X, Z) \), where \( S_Y = \{y \in Y : \|y\| = 1\} \) is the unit sphere in \( Y \). Now fix \( x \in X \). Since \( b(\cdot, \cdot) \) is continuous, there is a \( C \geq 0 \) such that \( \|b(x, y)\| \leq C\|y\| \) for all \( y \in Y \). In particular, this means that

\[
\sup_{y \in S_Y} \|b(x, y)\| \leq C.
\]

Thus, by the Uniform Boundedness Principle we have

\[
c := \sup_{y \in S_Y} \|b(\cdot, y)\| < \infty.
\]

Now let \( x \in X, y \in Y \). If \( y \neq 0 \), then \( \frac{y}{\|y\|} \in S_Y \) and

\[
\|b(x, y)\| = \|b(x, \frac{y}{\|y\|})\|\|y\| \leq \|b(\cdot, \frac{y}{\|y\|})\|\|x\|\|y\| \leq c\|x\|\|y\|.
\]

If \( y = 0 \), then this inequality also holds, proving (iii).

“(iii)⇒(i)”

Let \( (x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \) be sequences respectively in \( X, Y \) with respective limits \( x \in X, y \in Y \). Then

\[
\|b(x_n, y_n) - b(x, y)\| \leq \|b(x_n, y_n) - b(x_n, y)\| + \|b(x_n, y) - b(x, y)\| = \|b(x_n, y_n - y)\| + \|b(x_n - x, y)\| \leq c\|x_n\|\|y_n - y\| + c\|x_n - x\|\|y\| \leq c \left( \sup_{n \in \mathbb{N}} \|x_n\| \right) \|y_n - y\| + c\|x_n - x\|\|y\| \to 0
\]

as \( n \to \infty \). Thus, \( b \) is continuous, as asserted.