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Fourier Analysis

Exercise Sheet 1

The first exercise class is intended to present and prove the Stone-Weierstrass theorems. These are generalizations of the Weierstrass theorem, which allow for the approximation of continuous functions on a compact Hausdorff space by more general subalgebras than the space of polynomials. We begin by recalling needed definitions and facts.

Definition 1.9. *Let G be a collection of functions from S into $\mathbb{K}$, where S is a set and $\mathbb{K}$ is the field $\mathbb{R}$ or $\mathbb{C}$.*

(a) G separates points (of S) provided for all $x, y \in S$ with $x \neq y$ there is a function $g \in G$ such that $g(x) \neq g(y)$.

(b) G is an algebra (of functions over $\mathbb{K}$)[1] provided if $f, g \in G$ and $\alpha \in \mathbb{K}$, then $f + g$, fg, and αf belong to G.

Recall 1.10. *Let S be a compact Hausdorff[2] topological space. Let $\mathbb{K}$ be the field $\mathbb{R}$ or $\mathbb{C}$. Let $C(S, \mathbb{K})$ be the vector space (over $\mathbb{K}$) of all continuous functions from S into $\mathbb{K}$.*

$$C(S, \mathbb{K}) = \{f : S \to \mathbb{K} \mid f \text{ is continuous}\} \ .$$

We can (and will for this Exercise Sheet) equip $C(S, \mathbb{K})$ with its usual norm

$$\|f\|_\infty := \sup_{s \in S} |f(s)| \ ,$$

which generates the topology of uniform convergence on $C(S, \mathbb{K})$. Note that $C(S, \mathbb{K})$ separates points and is an algebra of functions. Also, $(C(S, \mathbb{K}), \|\cdot\|_\infty)$ is a Banach space.

Theorem 1.11 (Stone-Weierstrass theorem, real version so $\mathbb{K} = \mathbb{R}$).
*Let S be a compact Hausdorff topological space. Let G be a **subalgebra** of $C(S, \mathbb{R})$ having the following two properties.*

(a) G separates points.

(b) G contains the constant function 1_S (which is defined by $1_S(\cdot) = 1$).

Then G is dense in $C(S, \mathbb{R})$.

[1] under pointwise operations, as usual, so e.g., the function $f + g$ is defined by $(f + g)(\cdot) := (f(\cdot)) + (g(\cdot))$
[2] Let X be a topological space. X is Hausdorff provided for all $x, y \in X$ with $x \neq y$ there exists disjoint open sets U_x and U_y with $x \in U_x$ and $y \in U_y$.

— please turn over —

The proof of this theorem is our first exercise.

Exercise 1
Prove Theorem 1.11. For that purpose, proceed in the following steps. Do so without using Corollary 1.12, which is to come.

(a) Let $v : [0, 1] \to \mathbb{R}$ be given by $v(\cdot) = \sqrt{\cdot}$. Show that there exists an increasing sequence of polynomials that converges to v in $C([0, 1], \mathbb{R})$, i.e., converges to v uniformly.

(b) Conclude that $|f|$ belongs to $\overline{G}$ for each $f \in G$.

(c) Prove that for all functions $f, g \in \overline{G}$, $\inf\{f, g\}$ and $\sup\{f, g\}$ belong to $\overline{G}$.

(d) Let $\alpha, \beta \in \mathbb{R}$, $x, y \in E$ with $x \neq y$. Show that there is a function $f \in G$ with $f(x) = \alpha$ and $f(y) = \beta$.

(e) Let $f \in C(S, \mathbb{R})$, $x \in S$, and $\varepsilon > 0$. Show that there exists a function g in $\overline{G}$ with $g(x) = f(x)$ and $g(y) \leq f(y) + \varepsilon$ for all $y \in S$.

(f) Show that G is dense in $C(S, \mathbb{R})$.

The well-known Weierstrass Approximation Theorem follows directly from Theorem 1.11.

Corollary 1.12 (Weierstrass Approx. Theorem)**.** *The polynomials are dense in* $C([0, 1], \mathbb{R})$.

Exercise 2
Find an example of a nondense subalgebra G of $C([0, 1], \mathbb{R})$ that satisfies condition (a) but not condition (b) of Theorem 1.11.

Exercise 3
Find an example of a nondense subalgebra G of $C([0, 1], \mathbb{R})$ that satisfies condition (b) but not condition (a) of Theorem 1.11.

We also want to give a complex version of the Stone-Weierstrass theorem. Observe that we have the same prerequisites as in the real-valued case, additionally assuming that G is closed under complex conjugation.

Theorem 1.13 (Stone-Weierstrass theorem, complex version so $\mathbb{K} = \mathbb{C}$)**.**
Let S be a compact Hausdorff topological space. Let G be a ***subalgebra*** *of $C(S, \mathbb{C})$ having the following three properties.*

(a) G separates points.

(b) $1_S \in G$.

(c) $\overline{f}$ belongs to G for all $f \in G$.

Then G is dense in $C(S, \mathbb{C})$.

Exercise 4
Prove Theorem 1.13 by reducing it to Theorem 1.11.

Exercise 5
Let $S = \{z \in \mathbb{C} \colon |z| \leq 1\}$, endowed with the subspace topology from $\mathbb{C}$. Find an example of a nondense subalgebra G of $C(S, \mathbb{C})$ that satisfies conditions (a) and (b) but not condition (c) of Theorem 1.13.