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SS 2006

Problems for Harmonic Analysis

Please hand in your solutions before Wednesday, 10 May 2006, 11:30.

Problem 1.1

(a) Let $f, g \in L_1(\mathbb{T})$, $k \in \mathbb{Z}$, and $s \in \mathbb{T}$. Using the notation introduced in Section 1.2, show that

$$\widehat{\overline{f}}(k) = \overline{\widehat{f}(-k)}, \qquad \widehat{\widetilde{f}}(k) = \widehat{f}(-k), \qquad \widehat{\tau_s f}(k) = e^{-2\pi iks}\widehat{f}(k).$$

(b) Let $f \in C^n(\mathbb{T})$ and $n \in \mathbb{N}$. Prove that

$$\widehat{f^{(n)}}(k) = (2\pi ik)^n \widehat{f}(k), \qquad k \in \mathbb{Z}$$

Problem 1.2

Let $f \in L^1(\mathbb{T})$, let n be a positive integer, and write $(\delta_n f)(t) = f(nt)$. Compute the k-th Fourier coefficient $\widehat{\delta_n f}(k)$ of $\delta_n f$.

Hint: Consider the two cases $\frac{k}{n} \in \mathbb{Z}$ and $\frac{k}{n} \notin \mathbb{Z}$.

Problem 1.3

Let $1 \le p, q, r \le \infty$ satisfy $\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}$. Prove that if $f \in L^p(\mathbb{T})$ and $g \in L^r(\mathbb{T})$ then the convolution $f * g \in L^q(\mathbb{T})$ and

$$\|f * g\|_{L^q(\mathbb{T})} \le \|g\|_{L^r(\mathbb{T})} \|f\|_{L^p(\mathbb{T})}.$$

Hint: Write $|f|$, $|g|$ as products using $\frac{p}{q} + \frac{p}{r'} = 1$ and $\frac{r}{q} + \frac{r}{p'} = 1$. Then apply Hölder's inequality. (Observe that $\frac{1}{r'} + \frac{1}{q} + \frac{1}{p'} = 1$.)

Problem 1.4

The Poisson kernel on $\mathbb{T}$ is the function

$$P_r(t) = \sum_{k \in \mathbb{Z}} r^{|k|} e^{2\pi ikt}, \qquad t \in \mathbb{T}, r \in [0, 1).$$

(a) Prove that

$$P_r(t) = \mathrm{Re}\left(\frac{1 + re^{2\pi it}}{1 - re^{2\pi it}}\right) = \frac{1 - r^2}{1 - 2r\cos(2\pi t) + r^2}.$$

(b) Show that P_r is an approximate identity as $r \uparrow 1$.