Universität Karlsruhe
Mathematisches Institut I
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SS 2006

Problems for Harmonic Analysis

Please hand in your solutions before Wednesday, 24 May 2006, 11:30.

Problem 2.1

(i) Let (s_k) be a sequence in $\mathbb{C}$ and s a complex number. Prove that if $s_k \to s$ as $k \to \infty$, then $\frac{1}{n+1}\sum_{k=0}^{n} s_k \to s$ as $n \to \infty$.

(ii) Find a sequence (s_k) in $\mathbb{C}$ such that (s_k) does not converge, but $(\frac{1}{n+1}\sum_{k=0}^{n} s_k)$ does.

Problem 2.2

Let D_n be the Dirichlet kernel on $\mathbb{T}$. Prove that

$$\frac{4}{\pi^2}\sum_{k=1}^{2n}\frac{1}{k} \leq \|D_n\|_{L^1} \leq 1 + \frac{\pi}{4} + \frac{4}{\pi^2}\sum_{k=1}^{2n}\frac{1}{k}.$$

Conclude that the numbers $\|D_n\|_{L^1}$ grow logarithmically as $n \to \infty$ and therefore (D_n) is not an approximate identity on $\mathbb{T}$. The numbers $\|D_n\|_{L^1}$ are called **Lebesgue constants**.

Hint: Use that $|\frac{1}{\sin(\pi x)} - \frac{1}{\pi x}| \leq \frac{\pi}{4}$ when $|x| \leq \frac{1}{2}$.

Problem 2.3

Use the open mapping theorem (see Appendix A) to prove that there exists a sequence $(a_k) \in c_0$ for which there is no function $f \in L^1(\mathbb{T})$ with $\widehat{f}(k) = a_k$ for all $k \in \mathbb{Z}$.

Problem 2.4

(i) Assume that $f, g \in A(\mathbb{T})$. Prove that $fg \in A(\mathbb{T})$ and

$$\|fg\|_{A(\mathbb{T})} \leq \|f\|_{A(\mathbb{T})}\|g\|_{A(\mathbb{T})}.$$

(ii) Let $f, g \in L^2(\mathbb{T})$. Show that $f * g \in A(\mathbb{T})$.

(iii) Let f be differentiable on $\mathbb{T}$ with derivative $f' \in L^2(\mathbb{T})$. Prove that $f \in A(\mathbb{T})$ and

$$\|f\|_{A(\mathbb{T})} \leq \|f\|_{L^1} + \frac{1}{2\pi}\left(2\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{\frac{1}{2}}\|f'\|_{L^2}.$$

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