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SS 2006

Problems for Harmonic Analysis

Please hand in your solutions before Wednesday, 14 June 2006, 11:30.

Problem 3.1

Let P_r be the Poisson kernel on $\mathbb{T}$. For $r \in [0,1)$, define the **conjugate Poisson kernel** Q_r on $\mathbb{T}$ by

$$Q_r(t) = -i \sum_{k\in\mathbb{Z}} \operatorname{sign}(k) r^{|k|} e^{2\pi i k t}, \qquad t \in \mathbb{T}.$$

(a) Prove the identity

$$Q_r(t) = \frac{2r\sin(2\pi t)}{1 - 2r\cos(2\pi i) + r^2}.$$

(b) Let $f \in L^1(\mathbb{T})$. Show that the functions $z \mapsto (P_r * f)(t)$ and $z \mapsto (Q_r * f)(t)$ are harmonic functions of $z = re^{2\pi i t}$ in $\mathbb{D}$. Show that $z \mapsto (P_r*f)(t)+i(Q_r*f)(t)$ is analytic in $\mathbb{D}$.

Problem 3.2

(a) Prove that if $f \in L^1(\mathbb{R}^n)$, then $\widehat{f}$ is uniformly continuous on $\mathbb{R}^n$.

(b) Prove that for $f, g \in L^1(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^n} f(x)\widehat{g}(x)dx = \int_{\mathbb{R}^n} \widehat{f}(x)g(x)dx.$$

(c) Prove that if f and $\widehat{f}$ are both in $L^1(\mathbb{R}^n)$, then $\mathcal{F}^{-1}(\widehat{f}) = f$ a.e.

Problem 3.3 (Uncertainty principle)

(a) Prove that for $f \in \mathcal{S}(\mathbb{R})$

$$\|f\|_{L^2}^2 \le 4\pi \left(\int_{\mathbb{R}} |xf(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |\xi\widehat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}}.$$

Hint: Use integration by parts and the Cauchy-Schwarz inequality.

(b) Use (a) and the elementary properties of the Fourier transform to prove that for $f \in \mathcal{S}(\mathbb{R})$

$$\|f\|_{L^2}^2 \le 4\pi \inf_{y\in\mathbb{R}} \left(\int_{\mathbb{R}} |(x-y)f(x)|^2 dx\right)^{\frac{1}{2}} \inf_{\eta\in\mathbb{R}} \left(\int_{\mathbb{R}} |(\xi-\eta)\widehat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}}.$$

Problem 3.4

Let $\psi \in L^2(\mathbb{R})$ such that $\int_0^\infty |\widehat{\psi}(t)|^2 \frac{dt}{t} = \int_0^\infty |\widehat{\psi}(-t)|^2 \frac{dt}{t} = 1$. For $t > 0$ and $x \in \mathbb{R}$ we write $\psi_t(x) = \frac{1}{t}\psi(\frac{x}{t})$. Prove that for all $f \in \mathcal{S}(\mathbb{R})$

$$\int_{\mathbb{R}} \int_0^\infty |(\psi_t * f)(s)|^2 \frac{dt}{t} ds = \int_{\mathbb{R}} |f(x)|^2 dx.$$