Universität Karlsruhe
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SS 2006

Problems for Harmonic Analysis

Please hand in your solutions before Wednesday, 28 June 2006, 11:30.

Problem 4.1

(a) Prove that there exists a constant $B > 0$ such that for all $t > 0$

$$\left| \int_0^t \frac{\sin(\xi)}{\xi} d\xi \right| \leq B.$$

(b) If $f \in L^1(\mathbb{R})$ is odd, then for all $t > 0$

$$\left| \int_0^t \frac{\widehat{f}(\xi)}{\xi} d\xi \right| \leq B\|f\|_{L^1}.$$

(c) Let $g \in C(\mathbb{R})$ be odd with $g(\xi) = \frac{1}{\log(\xi)}$ for $\xi \geq 2$. Show that there does not exist a function $f \in L^1(\mathbb{R})$ such that $\widehat{f} = g$.

Problem 4.2

(a) For $-\infty < a < b < \infty$, compute the distributional derivative of the characteristic function $\chi_{[a,b]}$ of the interval $[a, b]$.

(b) Let $g(x) = \log |x|$, $x \in \mathbb{R} \setminus \{0\}$. Show that g is associated to a tempered distribution.

(c) Prove that the distributional derivative of g from (b) is v, given by

$$v(f) = \lim_{\delta \downarrow 0} \int_{\delta \leq |x|} f(x) \frac{dx}{x}, \qquad f \in \mathcal{S}(\mathbb{R}).$$

(d) Compute the Fourier transform of v from (c).
Hint: Use Problem 4.1 (a) and $\lim_{R\to\infty} \int_{-R}^{R} \frac{\sin(bx)}{x} dx = \pi \operatorname{sign}(b)$ for $b \in \mathbb{R}$.

Problem 4.3

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \varphi(x)dx = 1$ and set $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(\frac{x}{\varepsilon})$, $\varepsilon > 0$, $x \in \mathbb{R}^n$.

(a) Show that for all $f \in \mathcal{S}(\mathbb{R}^n)$, $\varphi_\varepsilon * f \to f$ in $\mathcal{S}(\mathbb{R}^n)$ as $\varepsilon \downarrow 0$.

(b) Conclude that for all $u \in \mathcal{S}'(\mathbb{R}^n)$, $\varphi_\varepsilon * u \to u$ in $\mathcal{S}'(\mathbb{R}^n)$ as $\varepsilon \downarrow 0$. Prove that in particular the tempered distributions associated to φ_ε converge to $\delta_{(0)}$ in $\mathcal{S}'(\mathbb{R}^n)$ as $\varepsilon \downarrow 0$.

Problem 4.4

Prove that if $f \in L^q(\mathbb{R}^n)$ and $1 \leq q < \infty$, then

$$\|\tau_b f + f\|_{L^q} \to 2^{1/q}\|f\|_{L^q} \qquad \text{as } |b| \to \infty.$$