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SS 2006

Problems for Harmonic Analysis

Please hand in your solutions before Wednesday, 12 July 2006, 11:30.

Problem 5.1

Let $m \in \mathcal{M}_p(\mathbb{R}^n)$ for some $p \in [1, 2]$.

(a) Prove that for $b \in \mathbb{R}^n$, $a > 0$, and A an orthogonal matrix in $\mathbb{R}^{n \times n}$, the functions $\tau_b m$, $\delta_a m$, $\widetilde{m}$, $e^{2\pi i (\cdot)\cdot b} m$, and $m \circ A$ are in $\mathcal{M}_p(\mathbb{R}^n)$ and

$$\|m\|_{\mathcal{M}_p} = \|\tau_b m\|_{\mathcal{M}_p} = \|\delta_a m\|_{\mathcal{M}_p} = \|\widetilde{m}\|_{\mathcal{M}_p} = \|e^{2\pi i (\cdot)\cdot b} m\|_{\mathcal{M}_p} = \|m \circ A\|_{\mathcal{M}_p}.$$

(b) Show that if $\psi \in L^1(\mathbb{R}^n)$, then $\psi * m \in \mathcal{M}_p(\mathbb{R}^n)$ and $\|\psi * m\|_{\mathcal{M}_p} \leq \|\psi\|_{L^1} \|m\|_{\mathcal{M}_p}$.

Problem 5.2

Let $p \in [1, 2]$.

(a) Suppose that $(m_k)_{k=1}^{\infty}$ is a sequence in $\mathcal{M}_p(\mathbb{R}^n)$ with $\|m_k\|_{\mathcal{M}_p} \leq C$ for all $k \in \mathbb{N}$ and $m_k \to m$ a.e. Prove that $m \in \mathcal{M}_p(\mathbb{R}^n)$ and

$$\|m\|_{\mathcal{M}_p} \leq \liminf_{k \to \infty} \|m_k\|_{\mathcal{M}_p} \leq C.$$

(b) Show that $(\mathcal{M}_p(\mathbb{R}^n), \|\cdot\|_{\mathcal{M}_p})$ is a Banach space.

(c) Suppose now that $m_t \in \mathcal{M}_p(\mathbb{R}^n)$ for all $t \in (0, \infty)$ and $\int_0^\infty \|m_t\|_{\mathcal{M}_p} \frac{dt}{t} < \infty$. Let $m(\xi) = \int_0^\infty m_t(\xi) \frac{dt}{t}$. Prove that $m \in \mathcal{M}_p(\mathbb{R}^n)$.

Problem 5.3

Let $p \in [1, 2]$, $(a_k)_{k \in \mathbb{Z}} \in \mathbb{C}$ and $A > 0$. Prove that the following are equivalent:

(a) $f \mapsto \sum_{k \in \mathbb{Z}} a_k f(\cdot - k)$ defines a bounded linear operator from $L^p(\mathbb{R})$ to $L^p(\mathbb{R})$ with norm equal to A.

(b) The function $\xi \mapsto \sum_{k \in \mathbb{Z}} a_k e^{-2\pi i k \xi}$ is in $\mathcal{M}_p(\mathbb{R})$ with norm A.

(c) $(x_j)_{j \in \mathbb{Z}} \mapsto (\sum_{k \in \mathbb{Z}} a_k x_{j-k})_{j \in \mathbb{Z}}$ defines a bounded linear operator from ℓ^p to ℓ^p with norm A.

Problem 5.4

(a) Show that the Hilbert transform H on $L^2(\mathbb{R})$ commutes with translations and dilations and anticommutes with the reflection, i.e., $H\widetilde{f} = -\widetilde{Hf}$.

(b) Prove that if $T \in \mathcal{L}(L^2(\mathbb{R}))$ commutes with translations and dilations and anticommutes with the reflection, then T is a constant multiple of the Hilbert transform.