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Project 1: Lacunary Series

Definition: A sequence $(\lambda_k)_{k=1}^{\infty}$ of positive integers is called **lacunary** if there exists a constant $A > 1$ such that $\lambda_{k+1} \geq A\lambda_k$ for all $k \in \mathbb{N}$.

Instructions:

1. Give examples of lacunary sequences.
2. Observe that lacunary sequences tend to infinity as $k \to \infty$. Give an example of a sequence that tends to infinity as $k \to \infty$ but is not lacunary.
3. Study the proof of Proposition 3.7.2 on p. 236-237 in [1] and fill in the details.
4. Using Proposition 3.7.2, prove the following:

   Let $1 < a, b < \infty$ and consider the 1-periodic function

   $$f(t) = \sum_{k=0}^{\infty} a^{-k} e^{2\pi i b^k t}.$$

   Then the following statements are equivalent:

   (i) f is differentiable everywhere.
   (ii) f is differentiable at a point.
   (iii) $a > b$.
5. Conclude that the Weierstrass function

   $$f(t) = \sum_{k=0}^{\infty} 2^{-k} e^{2\pi i 3^k t}$$

   is continuous, but nowhere differentiable.

References and further reading:

[1] L. Grafakos: Classical and Modern Fourier Analysis, 2004, p. 235-244.

[2] Y. Katznelson: An Introduction to Harmonic Analysis, 3rd ed. 2004, p. 133-142.

[3] T. W. Körner: Fourier Analysis, 1989.