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Project 2: Decay of Fourier Coefficients

Definition: A sequence $(a_k)_{k=0}^{\infty}$ of positive real numbers is called **convex** if

$$a_{k+1} + a_{k-1} - 2a_k \geq 0 \qquad \text{for all } k \in \mathbb{N}.$$

Instructions:

1. Let (a_k) be a convex decreasing sequence of positive real numbers with $\lim_{k\to\infty} a_k = 0$ and let $m \geq 0$ be an integer. Prove that
$$\sum_{k=1}^{\infty} k(a_{k+m+1} + a_{k+m-1} - 2a_{k+m}) = a_m.$$
2. Let (a_k) be as in 1. Show that there exists a function $f \in L^1(\mathbb{T})$ such that $\widehat{f}(k) = a_{|k|}$ for all $k \in \mathbb{Z}$.
Hint: Try $f = \sum_{k=1}^{\infty} k(a_{k+1} + a_{k-1} - 2a_k)F_{k-1}$.
3. Given a sequence (c_k) of positive real numbers that tends to zero, show that there is a decreasing convex sequence satisfying $a_k \geq c_k$ and $a_k \downarrow 0$.
4. Conclude that given a sequence (c_k) of positive real numbers that tends to zero, there is a function $f \in L^1(\mathbb{T})$ such that $|\widehat{f}(k)| \geq c_k$ for all $k \in \mathbb{Z}$. In other words, given any rate of decay, there is an integrable function on $\mathbb{T}$ whose Fourier coefficients have slower rate of decay.

References and further reading:

[1] L. Grafakos: Classical and Modern Fourier Analysis, 2004, p. 172-175.

[2] Y. Katznelson: An Introduction to Harmonic Analysis, 3rd ed. 2004, p. 23-25.

[3] T. W. Körner: Fourier Analysis, 1989.

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