Universität Karlsruhe
Mathematisches Institut I
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Project 3: The Poisson kernel on $\mathbb{R}^n$

Definition: Let

$$\mathcal{P}(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \frac{1}{(1+|x|^2)^{\frac{n+1}{2}}}, \qquad x \in \mathbb{R}^n,$$

where Γ is the Gamma function given by $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$, $\operatorname{Re} z > 0$. The **Poisson kernel** $\mathcal{P}_t$ on $\mathbb{R}^n$ is now defined by $\mathcal{P}_t(x) = t^{-n}\mathcal{P}(\frac{x}{t})$ for $t > 0$ and $x \in \mathbb{R}^n$.

Instructions:

1. For $f \in L^1(\mathbb{R})$, prove that $\int_{\mathbb{R}} f(s)ds = \int_{\mathbb{R}} f(s - \frac{1}{s})ds$.
2. Use 1. with $f(s) = e^{-rs^2}$ to obtain
$$e^{-2r} = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-y-\frac{r^2}{y}} \frac{dy}{\sqrt{y}}, \qquad r > 0.$$
Set $r = \pi|x|$, $x \in \mathbb{R}^n$, and integrate with respect to $e^{-2\pi i\xi\cdot x}dx$ to prove that $\mathcal{F}(e^{-2\pi|\cdot|}) = \mathcal{P}$.
3. Show that $\mathcal{P}_t$ is an approximate identity as $t \downarrow 0$.
4. For $f \in \mathcal{S}(\mathbb{R}^n)$ define
$$u(t,x) = \begin{cases} (\mathcal{P}_t * f)(x), & t > 0 \\ f(x), & t = 0. \end{cases}$$
Show that $u \in C^2((0,\infty)\times\mathbb{R}^n) \cap C([0,\infty)\times\mathbb{R}^n)$ and solves the Dirichlet problem
$$\begin{cases} u_{tt}(t,x) + \Delta_x u(t,x) &= 0, \qquad (t,x) \in (0,\infty)\times\mathbb{R}^n, \\ u(0,x) &= f(x), \qquad x \in \mathbb{R}^n. \end{cases}$$
5. Show that the Poisson kernel satisfies the semigroup property $\mathcal{P}_t * \mathcal{P}_s = \mathcal{P}_{t+s}$, $t, s > 0$.
6. Now let $n = 1$. Show that for $s \in \mathbb{R}$, $t > 0$, and $r = e^{-2\pi t}$ we have that
$$\sum_{k\in\mathbb{Z}} \mathcal{P}_t(s+k) = P_r(s),$$
where P_r is the Poisson kernel on $\mathbb{T}$.

References and further reading:

[1] E.M. Stein, R. Shakarchi: Fourier Analysis, 2003, pp. 149-151; 157-158.

[2] E.M. Stein: Singular Integrals and Differentiability Properties of Functions, 1970, pp. 60-65.