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Nichtlineare Evolutionsgleichungen
1. Übungsblatt

Problem 1
Let A generate the C_0-semigroup $T(\cdot)$ on a Hilbert space X, F belong to $C^1_{\mathbb{R}}(X) = C^1_{\mathbb{R}}(X, X)$, and let F' be bounded on bounded sets. We assume that $\mathrm{Re}(Ax + F(x)|x) \leq \omega \|x\|$ for some $\omega \in \mathbb{R}$ and all $x \in \mathrm{D}(A)$. Let $u_0 \in X$ and u be the corresponding maximal mild solution of $u'(t) = Au(t) + F(u(t))$ for $t \in [0, t^+(u_0))$ with $u(0) = u_0$.

Show that $\|u(t)\| \leq \mathrm{e}^{\omega t} \|u_0\|$ for all $t \in [0, t^+(u_0))$ and deduce that $t^+(u_0) = \infty$ for every $u_0 \in X$.

(Hint: One can use Theorem 8.10 of the manuscript of the Internet Seminar 2012/13, which will be proved in the lecture of 13 May.)

Problem 2
Let A generate the C_0-semigroup $T(\cdot)$, $J = [0, T]$, and $F : J \times X \to X$ be continuous. Assume that there is a constant $L > 0$ such that

$$\|F(t, x) - F(t, y)\| \leq L\|x - y\|$$

for all $x, y \in X$ and $t \in J$. Let $u_0 \in X$ and $s \in J$. Show that there is a unique solution $u = u(\,\cdot\,; s, u_0) \in C([s, T], X)$ of the equation

$$u(t) = T(t - s)u_0 + \int_s^t T(t - \tau)F(\tau, u(\tau))\,\mathrm{d}\tau, \quad t \in [s, T].$$

In addition, let $x \mapsto F(s, x)$ be linear for each $s \in [0, T]$. We set $U(t, s)u_0 = u(t; s, u_0)$ for $0 \leq s \leq t \leq T$ and $u_0 \in X$. Show that

i) $U(t, s) \in \mathcal{B}(X)$ and $\sup_{0 \leq s \leq t \leq T} \|U(t, s)\| < \infty$;

ii) $U(t, t) = I$ and $U(t, r)U(r, s) = U(t, s)$ for $0 \leq s \leq r \leq t \leq T$;

iii) the map $\{(t, s) \,|\, 0 \leq s \leq t \leq T\} \to X,; (t, s) \mapsto U(t, s)x$, is continuous for all $x \in X$.

Problem 3
As in Example 1.48 of *Evolution Equations* one can prove that the operator A given by $Au = \partial_x^2 u$ with $\mathrm{D}(A) = \{u \in C^2([0, 1]) \,|\, \partial_x u(0) = \partial_x u(1) = 0\}$ generates a C_0–semigroup on $X = C([0, 1])$. Let $\varphi \in C^1(\mathbb{R}, \mathbb{R})$ and set $F(u) := \varphi(\mathrm{Re}\, u)$ for all $u \in X$. Show the following assertions.

a) $F : X \to X$ is Lipschitz on balls.

b) The 'reaction-diffusion equation'

$$u'(t) = Au(t) + F(u(t)), \quad t \in J, \qquad u(0) = u_0,$$

has for all $u_0 \in X$ a unique maximal mild solution $u \in C([0, t^+(u_0)), X)$. If u_0 is real valued, then also u is real valued.

c) Let $\varphi(s) = s^2$. Find an initial function $u_0 \in X$ such that $t^+(u_0) < \infty$.