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17. Mai 2019
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Nichtlineare Evolutionsgleichungen
3. Übungsblatt

Problem 1
Let the standing wave $w(t,x) = e^{i\omega t} v(x)$ solve the semilinear wave equation (1.14) for some $\omega \in \mathbb{R}$. Derive the equation that v has to solve.

Problem 2
Let $\alpha > 1$ and $a \in \mathbb{R}$. Let $w \in C^2(\mathbb{R}_{\geq 0}, L^2(\mathbb{R}^3)) \cap C^1(\mathbb{R}_{\geq 0}, W^{1,2}(\mathbb{R}^3)) \cap C(\mathbb{R}_{\geq 0}, W^{2,2}(\mathbb{R}^3))$ solve the nonlinear wave equation

$$\partial_{tt} w(t) = \Delta w(t) - a\,|w(t)|^{\alpha-1} w(t), \quad t \geq 0, \qquad w(0) = w_0, \quad \partial_t w(0) = w_1, \tag{1}$$

on $\mathbb{R}^3$. Let $\lambda, \beta, \gamma, \delta > 0$ and define the rescaled function $w_\lambda(t,x) = \lambda^\beta w(\lambda^\gamma t, \lambda^\delta x)$ for $t \geq 0$ and $x \in \mathbb{R}^3$. For which parameters β, γ, and δ the function w_λ solves the differential equation in (1) for each $\alpha > 0$ and what are the resulting initial data?

These parameters depend on α. We now take $\gamma = 1$. The energy of w is

$$E_w(t) = \tfrac{1}{2}\,\|\nabla w(t)\|_2^2 + \tfrac{1}{2}\,\|\partial_t w(t)\|_2^2 + \tfrac{a}{\alpha+1}\,\|w(t)\|_{\alpha+1}^{\alpha+1}.$$

For which $\alpha > 1$ we have $E_{w_\lambda}(t) = E_w(\lambda t)$ for all $t \geq 0$ and $\lambda > 0$?

Problem 3
Let $w \in C^2(\mathbb{R} \times \mathbb{R}^3)$ solve (1) pointwise. Take fixed $\theta, t_0 \in \mathbb{R}$, $x_0, v \in \mathbb{R}^3$ with $|v|_2 \in (0,1)$, and orthogonal $Q \in \mathbb{R}^{3\times 3}$. Set $\gamma = (1 - |v|_2^2)^{-1/2}$ and $x_v = (x \cdot v)|v|_2^{-2}\, v$ for $x \in \mathbb{R}^3$. Show that also the functions w_k on $\mathbb{R} \times \mathbb{R}^3$ given by

$$w_1(t,x) = e^{i\theta} w(t - t_0, x - x_0), \qquad w_2(t,x) = w(-t,x), \qquad w_3(t,x) = \overline{w(t,x)},$$
$$w_4(t,x) = w(t, Qx), \qquad w_5(t,x) = w\big(\gamma(t - v\cdot x), x - x_v + \gamma(x_v - vt)\big)$$

solve (1) with modified initial data.

Problem 4
Let w be as in Problem 2 with $a \geq 0$ and assume that $w_0 = w_1 = 0$ on $\overline{B}(0,r)$ for some $r > 0$. Show that $w(t,x) = 0$ on the cone $C(r) = \{(t,x) \,|\, t \in [0,r],\ |x| \leq r - t\}$.

(Hint: Note that $W^{2,2}(\mathbb{R}^3) \hookrightarrow C_0(\mathbb{R}^3)$. Set $B_t = B(0, r-t)$ and differentiate

$$e(t) = \int_{B_t} \Big(\frac{1}{2}\,|\partial_t w(t)|^2 + \frac{1}{2}\,|\nabla w(t)|^2 + \frac{a}{\alpha+1}\,|w(t)|^{\alpha+1}\Big) dx, \qquad t \in [0,r),$$

using $\frac{d}{dt}\int_{B_t} f(x,t)\,dx = \int_{B_t} (\partial_t f)(x,t)\,dx - \int_{\partial B_t} f(x,t)\,d\sigma$ for $t \in J$ if $f \in C^1(J, L^2_{\mathrm{loc}}(\mathbb{R}^3)) \cap C(J, W^{1,2}_{\mathrm{loc}}(\mathbb{R}^3))$.)