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Nichtlineare Evolutionsgleichungen

4. Übungsblatt

Problem 1
Let $q \in [1,\infty)$ and consider the translation semigroup $T(\,\cdot\,)$ on $X = L^q(\mathbb{R})$ as in Example 2.2 of the Lecture Notes on Asymptotics of Evolution Equations. Recall that $T(\,\cdot\,)$ has generator $A = d/ds$ with domain $D(A) = W^{1,p}(\mathbb{R})$ and that, given $p \in [1,\infty)$ and $\alpha \in (0,1)$, the norm on $D_A(\alpha,p)$ is given by

$$\|f\| \eqsim \|f\|_{L^q(\mathbb{R})} + \left(\int_{\mathbb{R}} |t|^{-\alpha p-1} \|f(\,\cdot\, + t) - f\|_{L^q(\mathbb{R})}\,\mathrm{d}t\right)^{1/p}$$

Let $I = (a,b)$ with $-\infty < a < b < \infty$. Show that $1_I \in D_A(\alpha,p)$ if and only if $\alpha < 1/q$.

Hint: Show that the symmetric difference $(I-t)\Delta I$ has Lebesgue measure $|(I-t)\Delta I| = 2(|t|1_{|t|<b-a} + (b-a)1_{|t|\geq b-a})$. Recall here that the symmetric difference of two sets A and B is given by $A\Delta B = (A\setminus B)\cup(B\setminus A) = (A\cup B)\setminus(A\cap B)$.

Problem 2
Let $p \in (1,\infty]$. Derive the following special case of Hardy's inequality:

$$\left\| t \mapsto \frac{1}{t}\int_0^t f(s)\,\mathrm{d}s \right\|_{L^p(0,1;X)} \leq \frac{p}{p-1}\|f\|_{L^p(0,1;X)}.$$

Problem 3
Let A generate the C_0-semigroup $T(\,\cdot\,)$ on X with generator A, put $M_0 := \sup_{t\in[0,1]}\|T(t)\|$ and let $p \in (1,\infty)$. Show that $u \mapsto u(0)$ is a bounded from $W^{1,p}(0,1;X)\cap L^p(0,1;D(A))$ to $D_A(1-1/p,p)$. The Sobolev space $W^{1,p}(0,1;X)$ is discussed on p.125–127 of the Internet Seminar Leture Notes.

Hint: Write $u(0) = -\int_0^t u'(s)\,\mathrm{d}s + u(t)$, use the formula $T(t)b - b = \int_0^t T(s)Ab\,\mathrm{d}s$ for $b \in D(A)$, and use the inequality from Problem 2.