Problem 1
Let \( q \in [1, \infty) \) and consider the translation semigroup \( T(\cdot) \) on \( X = L^q(\mathbb{R}) \) as in Example 2.2 of the Lecture Notes on Asymptotics of Evolution Equations. Recall that \( T(\cdot) \) has generator \( A = \frac{d}{ds} \) with domain \( D(A) = W^{1,p}(\mathbb{R}) \) and that, given \( p \in [1, \infty) \) and \( \alpha \in (0, 1) \), the norm on \( D_A(\alpha, p) \) is given by

\[
\|f\| \overset{\text{def}}{=} \|f\|_{L^q(\mathbb{R})} + \left( \int_{\mathbb{R}} |t|^{-\alpha p - 1} \|f(\cdot + t) - f\|_{L^q(\mathbb{R})} dt \right)^{1/p}
\]

Let \( I = (a, b) \) with \(-\infty < a < b < \infty\). Show that \( \chi_I \in D_A(\alpha, p) \) if and only if \( \alpha < 1/q \).

Hint: Show that the symmetric difference \( (I - t) \Delta I \) has Lebesgue measure \( |(I - t) \Delta I| = 2(\int_{|t|<b-a} + (b - a)\int_{|t|\geq b-a}) \). Recall here that the symmetric difference of two sets \( A \) and \( B \) is given by \( A \Delta B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B) \).

Problem 2
Let \( p \in (1, \infty] \). Derive the following special case of Hardy’s inequality:

\[
\left\| t \to \frac{1}{t} \int_0^t f(s) \, ds \right\|_{L^p(0,1;X)} \leq \frac{p}{p-1} \|f\|_{L^p(0,1;X)}.
\]

Problem 3
Let \( A \) generate the \( C_0 \)-semigroup \( T(\cdot) \) on \( X \) with generator \( A \), put \( M_0 := \sup_{t \in [0,1]} \|T(t)\| \) and let \( p \in (1, \infty) \). Show that \( u \to u(0) \) is a bounded from \( W^{1,p}(0,1;X) \cap L^p(0,1;D(A)) \) to \( D_A(1 - 1/p, p) \). The Sobolev space \( W^{1,p}(0,1;X) \) is discussed on p.125–127 of the Internet Seminar Lecture Notes.

Hint: Write \( u(0) = -\int_0^t u'(s) \, ds + u(t) \), use the formula \( T(t)b - b = \int_0^t T(s)Ab \, ds \) for \( b \in D(A) \), and use the inequality from Problem 2.