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Nichtlineare Evolutionsgleichungen
5. Übungsblatt

Problem 1

Assume A generates an analytic C_0-semigroup $T(\,\cdot\,)$ on X. Let $\alpha \in (0,1)$, $f \in C^\alpha([0,T],X)$ and $x \in D(A)$. Let u be the unique classical solution of

$$u'(t) = Au(t) + f(t), \qquad t \in [0,T], \qquad u(0) = x. \tag{1}$$

Show that, if $Ax + f(0) \in D_A(\alpha)$, then $u', Au \in C^\alpha([0,T],X)$ and $u' \in B([0,T],D_A(\alpha))$ with

$$\begin{aligned}&\|u\|_{C^{1+\alpha}([0,T],X)} + \|Au\|_{C^\alpha([0,T],X)} + \|u'\|_{B([0,T],D_A(\alpha)}\\ &\lesssim \|f\|_{C^\alpha([0,T],X)} + \|x\|_{D(A)} + \|Ax + f(0)\|_{D_A(\alpha)}.\end{aligned}$$

Hint: consider the splitting $u = u_1 + u_2$, where

$$\begin{aligned}u_1(t) &= \int_0^t T(t-s)(f(s) - f(t))\,\mathrm{d}s,\\ u_2(t) &= T(t)x + \int_0^t T(t-s)f(t)\,\mathrm{d}s.\end{aligned}$$

Problem 2

Assume A generates an analytic C_0-semigroup $T(\,\cdot\,)$ on X. Let $\alpha \in (0,1)$, $x \in D(A)$ and $f \in C([0,T],X) \cap B([0,T],D_A(\alpha))$. Let u be the unique classical solution of (1). Show that, if $Ax \in D_A(\alpha)$, then $u', Au \in C([0,T],X) \cap B([0,T],D_A(\alpha))$ and $Au \in C^\alpha([0,T],X)$ with

$$\begin{aligned}&\|u'\|_{B([0,T],D_A(\alpha))} + \|Au\|_{B([0,T],D_A(\alpha))} + \|Au\|_{C^\alpha([0,T],X)}\\ &\lesssim \|f\|_{B([0,T],D_A(\alpha)} + \|x\|_{D(A)} + \|Ax\|_{D_A(\alpha)}.\end{aligned}$$

Problem 3

Let Ω be a locally compact space and $m : \Omega \to \mathbb{C}$ a continuous function satisfying $\sup_{x\in\Omega} \Re(m(x)) < \infty$. Let $X = C_0(\Omega)$ and $T(\,\cdot\,)$ the C_0-semigroup on X generated by the multiplication operator

$$D(A) = \{u \in X : mu \in X\}, \quad Au = mu.$$

Show that, for each $\alpha \in (0,1)$,

$$D_A(\alpha) = \{u \in X : |m|^\alpha u \in X\}.$$