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Nichtlineare Evolutionsgleichungen
6. Übungsblatt

For a Banach space Y and $\theta, \delta \in (0,\infty)$, we define the space $C_\theta((0,\delta];Y)$ by

$$C_\theta((0,\delta];Y) := \{u \in C((0,\delta]) : \|u\|_{C_\theta((0,\delta];Y)} = \sup_{0<t\leq\delta} \|t^\theta u(t)\|_Y < \infty\}.$$

Problem 1
Let A generate an analytic C_0-semigroup $T(\,\cdot\,)$ on X. Let $a \in (0,T]$ and let $\varphi \in C_\theta((0,a];X)$ with $\theta \in (0,1)$. Define the function v by $v(t) := \int_0^t T(t-s)\varphi(s)\,\mathrm{d}s$. Show that $v \in C^{(1-\theta)}([0,a];X) \cap C_{\alpha+\theta-1}((0,a];X_\alpha)$ with

$$\|v\|_{C^{(1-\theta)}([0,a];X)} + \|v\|_{C_{\alpha+\theta-1}((0,a];X_\alpha)} \lesssim_{\alpha,\theta} \|\varphi\|_{C_\theta((0,a];X)}.$$

Hint: Defining the constants

$$M_k := \sup_{0<\leq t\leq T+1} t^k \|A^k T(t)\|_{L(X)}, \qquad k \in \mathbb{N},$$

and

$$K_{k,\alpha} := \sup_{0<\leq t\leq T+1} t^{k+\alpha} \|A^k T(t)\|_{L(X,D_A(\alpha,1))}, \qquad k \in \mathbb{N},$$

the term $\|v\|_{C^{(1-\theta)}([0,a];X)}$ can be estimated in terms of θ, M_0, M_1 and the term $\|v\|_{C_{\alpha+\theta-1}((0,a];X_\alpha)}$ can be estimated in terms of $\alpha, \theta, K_{0,\alpha}$.

Problem 2
Let A generate an analytic C_0-semigroup $T(\,\cdot\,)$ on X. Let $\alpha \in (0,1)$, $\gamma \in [1, 1/\alpha)$ and $C \in [0,\infty)$. Let $f : X_\alpha \to X$ be a continuous function such that

$$\|f(x) - f(y)\| \leq C\left[(1 + \|x\|_{X_\alpha}^{\gamma-1} + \|y\|_{X_\alpha}^{\gamma-1})\|x-y\|_{X_\alpha} + (\|x\|_{X_\alpha}^{\gamma} + \|y\|_{X_\alpha}^{\gamma})\|x-y\|\right]$$

for all $x, y \in X_\alpha$. Show that, for every $x \in X$ and $r \in (0,\infty)$, there exists $\delta, K \in (0,\infty)$ such that, for each $u_0 \in X$ with $\|u_0 - x\| \leq r$, the equation

$$u'(t) = Au(t) + f(u(t)), t > 0, \qquad u(0) = u_0,$$

has a unique mild solution $u = u(\,\cdot\,;u_0) \in C_\alpha((0,\delta];X_\alpha) \cap C([0,\delta];X)$; moreover, for each $u_0, u_1 \in X$ with $\|u_0 - x\|, \|u_1 - x\| \leq r$,

$$\|u(\,\cdot\,;u_0) - u(\,\cdot\,;u_1)\|_{C_\alpha((0,\delta];X_\alpha)} + \|u(\,\cdot\,;u_0) -= u(\,\cdot\,;u_1)\|_{C([0,\delta];X)} \leq K\|u_0 - u_1\|.$$

Hint: Using Problem 1 with $\theta = \alpha\gamma$ and Proposition 2.12(b) (from the Lecture Notes on Asymptotics of Evolution Equations) with $\beta = 0$, $k = 0$, do a fixed point argument in the set

$$B = B_{\delta,R} = \{u \in C_\alpha((0,\delta];X_\alpha) \cap C([0,\delta];X) : \|u - x\|_{C([0,\delta];X)} \leq r, \|u\|_{C_\alpha((0,\delta];X_\alpha)} \leq R\}.$$

— bitte wenden —

Problem 3
Let $\{a_{i,j}\}_{i,j=1,\ldots,m} \in \mathbb{R}^{m\times m}$ be symmetric and satisfy

$$\sum_{i,j=1}^{m} a_{i,j}\xi_i\xi_j \geq \nu|\xi|^2, \qquad \xi \in \mathbb{R}^m,$$

for some $\nu \in (0,\infty)$. Consider the operator A on $L^2(\mathbb{R}^m)$ with domain $D(A) = W^{2,2}(\mathbb{R}^m)$ given by $Au = \sum_{i,j} a_{i,j}\partial_{i,j}u$. Show that A generates an analytic C_0-semigroup $T(\,\cdot\,)$.

Hint: Use the Fourier transform.