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Nichtlineare Evolutionsgleichungen

7. Übungsblatt

Problem 1
Let $G \subset \mathbb{R}^m$ be bounded and open with $\partial G \in C^2$, $a \in (0,\infty)$, let $f \in C^1(\mathbb{R},\mathbb{R})$ and let u and v be real-valued functions that belong to $C([0,T]\times\overline{G}) \cap C^1((0,T],C(\overline{G})) \cap C((0,T];W^{2,q}(G))$ for all $q \in (1,\infty)$ and satisfy $\Delta u, \Delta v \in C((0,T]\times\overline{G})$. Assume that $\partial_t u - a\Delta u - f(u) \leq \partial_t v - a\Delta v - f(v)$ on $(0,T]\times\overline{G}$, $\partial_\nu u \leq \partial_\nu v$ on $(0,T]\times\partial G$ and that $u(0,\,\cdot\,) \leq v(0,\,\cdot\,)$. Show that $u \leq v$ on $[0,T]\times\overline{G}$.

Hint: consider $w := v - u$ and $\tilde{w}(t,x) := e^{c_0 t} w(t,x)$ for a suitable $c_0 \in \mathbb{R}$.

Problem 2
Let $G \subset \mathbb{R}^m$ be bounded and open with $\partial G \in C^2$, $a \in \mathbb{R}_+$ and let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = \Re(z)(1-\Re(z))$. Let $p \in (\frac{m}{2},\infty)$, $\alpha \in (\frac{m}{2p}, 1-\frac{m}{2p})$, $A = a\Delta_N$ on $L^p(G)$, and $u_0 \in D_A(\alpha,p)$ with $0 \leq u_0 \leq 1$. Show that $t^+(u_0) = \infty$ in the notation of Proposition 3.9 from the lecture.

Hint: use Problem 1.

Problem 3
Let $G \subset \mathbb{R}^m$ be bounded and open with $\partial G \in C^2$, let $p \in (\frac{m}{2},\infty)$, and let $\alpha \in (\frac{m}{2p}, 1-\frac{m}{2p})$. Consider the Dirichlet Laplacian $A = \Delta_D$ on $L^p(G)$ given by

$$D(\Delta_D) = \{v \in W^{2,p}(G) : v_{|\partial G} = 0\},\ \ \Delta_D v = \Delta v.$$

Let $u_0 \in D_A(\alpha,p)$ and let u be the unique mild solution of

$$u'(t) = \Delta_D u(t) + [u(t)]^2, t \in J, \quad u(0) = u_0,$$

with maximal existence time $t^+(u_0)$. It is known that there exists an eigenfunction w_1 corresponding to the smallest eigenvalue $\lambda_1 > 0$ of $-\Delta_D$ that belongs to $W^{2,q}(G)\cap W_0^{1,q}(G)$ for all $q \in (1,\infty)$ such that $w_1 > 0$ in G and $\int_G w_1\,\mathrm{d}x = 1$. Show that, if

$$\int_G u_0 w_1\,\mathrm{d}x > \lambda_1,$$

then $t^+(u_0) < \infty$.

Hint: consider

$$\eta(t) := \int_G u(x,t) w_1(x)\,\mathrm{d}x.$$