Nichtlineare Evolutionsgleichungen

8. Übungsblatt

Problem 1
Let A generate a C_0-semigroup $T(\cdot)$ on X, let $\alpha \in (0,1)$ and let $F : X_\alpha \to X$ be locally Lipschitz. Assume there exists an increasing function $\mu : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

$$\|F(x)\|_X \leq \mu(\|x\|_X)(1 + \|x\|_{X_\alpha}^\gamma), \qquad x \in X_\alpha,$$

for some $\gamma \in (1, \frac{1}{\alpha})$. Let $u_0 \in X_\alpha$ and let u be unique mild solution of

$$u'(t) = Au(t) + F(u(t)), t \in J, \quad u(0) = u_0,$$

with maximal existence time $t^+(u_0)$. Show that, if u is bounded on $[0, t^+(u_0))$ with values in X, then it is also bounded on $[0, t^+(u_0))$ with values in X_α.

Problem 2
Let B be a densely defined sectorial operator of angle $> \frac{3\pi}{4}$ on a Banach space X. Show that $A = -B^2$ is densely defined sectorial of angle $> \frac{\pi}{2}$. Moreover, show that $D(B) \hookrightarrow D_A(\frac{1}{2})$.

Problem 3
Let $f \in C^3(\mathbb{R}, \mathbb{R})$ and let $u_0 \in C^2([0,1], \mathbb{R})$ satisfy $u_0'(0) = u_0'(1) = 0$. Consider the equation

$$\begin{cases} \partial_t u(t,x) = \partial_x^2(-\partial_x^2 u(t,x) + f(u(t,x))), & t \in J, x \in [0,1], \\ \partial_x u(t,0) = \partial_x u(t,1) = \partial_x^3 u(t,0) = \partial_x^3 u(t,1), & t \in J, \\ u(0,x) = u_0(x), & x \in [0,1]. \end{cases} \tag{1}$$

Show that (1) has a unique solution $u \in C([0,\tau); C^2([0,1])) \cap C^1((0,\tau); C([0,1])) \cap C((0,\tau); C^4([0,1]))$ for some maximal $\tau > 0$. Moreover, show that, if f has a nonnegative primitive Φ, then $\tau = \infty$.

Hint: for the second statement you are allowed to use the inequality

$$\|\phi - \int_0^1 \phi(y)\,\mathrm{d}y\|_{C([0,1])} \leq \|\phi'\|_{L^2([0,1])}$$

for $\phi \in C^1([0,1])$.