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Nichtlineare Evolutionsgleichungen
9. Übungsblatt

Problem 1
Let $\alpha \in (1,\infty)$, let $\varphi \in H^2(\mathbb{R}^m)$ and let $u \in C(J, H^2(\mathbb{R}^m)) \cap C^1(J, L^2(\mathbb{R}^m))$ be a solution of

$$u'(t) = \mathrm{i}\Delta u(t) - \mathrm{i}\mu|u(t)|^{\alpha-1},\ t \in J, \quad u(0) = \varphi. \tag{1}$$

For $\lambda > 0$ and $\gamma, \kappa \in \mathbb{R}$ define the function $(u_\lambda(t))(x) = \lambda^\kappa(u(\lambda^\gamma t))(\lambda x)$, $t \in J$, $x \in \mathbb{R}^m$. For which exponent (γ, κ) does the map u_λ solve (1) for the initial value φ_λ given by $\varphi_\lambda(x) = \lambda^\kappa \varphi(\lambda x)$? Fix this exponent $(\gamma, \kappa) = (\overline{\gamma}, \overline{\kappa})$ in the definition of u_λ. For which $\alpha > 1$ we then have $\|u_\lambda(t)\|_2 = \|u(\lambda^2 t)\|_2$ or $\|\partial_k u_\lambda(t)\|_2 = \|\partial_k u(\lambda^2 t)\|_2$ for all $t \in J$ and $k \in \{1, \dots, m\}$?

Problem 2
Let $u \in C(\mathbb{R}, H^2(\mathbb{R}^m)) \cap C^1(\mathbb{R}, L^2(\mathbb{R}^m))$ be a solution of

$$u'(t) = \mathrm{i}\Delta u(t) - \mathrm{i}\mu|u(t)|^{\alpha-1},\ t \in \mathbb{R}, \quad u(0) = u_0. \tag{2}$$

For $h \in \mathbb{R}^m$, $Q \in \mathbb{R}^{m\times m}$ and $\varphi : \mathbb{R}^m \to \mathbb{C}$ we set $(S_h\varphi)(x) = \varphi(x-h)$ and $(R_Q\varphi)(x) = \varphi(Qx)$, $x \in \mathbb{R}^m$. We define

$$\begin{aligned}
w_1(t) &= \mathrm{e}^{\mathrm{i}\theta} S_{x_0} u(t - t_0) && \text{for fixed } t_0 \in \mathbb{R}, x_0 \in \mathbb{R}^m \text{ and } \theta \in \mathbb{R},\\
w_2(t) &= \overline{u(-t)},\\
w_3(t) &= R_Q u(t) && \text{for a fixed orthogonal } Q \in \mathbb{R}^{m\times m},\\
w_4(t) &= \mathrm{e}_{\mathrm{i}v} \mathrm{e}^{-\mathrm{i}|v|^2 t} S_{2vt} u(t) && \text{for a fixed } v \in \mathbb{R}^m,
\end{aligned}$$

where $t \in \mathbb{R}$ and $\mathrm{e}_{\mathrm{i}v}(x) = \mathrm{e}^{\mathrm{i}v\cdot x}$. Show that the functions w_j $(j = 1, 2, 3, 4)$ satisfy (2) for the appropriate initial values. Further show that $u(t)$ is spherically symmetric for all $t \in \mathbb{R}$ if $u(0)$ is spherically symmetric.

Problem 3
Let $p \in (2, \frac{2m}{m-2})$ and $r, s \in (1, \infty)$ such that

$$\frac{1}{r} + \frac{1}{s} = \frac{m}{2} - \frac{m}{p}.$$

Let $f \in L^{s'}(\mathbb{R}, L^{p'}(\mathbb{R}^m))$. Show that $T *_+ f \in L^r(\mathbb{R}, L^p(\mathbb{R}^m))$ and $\|T *_+ f\|_{L^r(\mathbb{R}, L^p(\mathbb{R}^m))} \le c\|f\|_{L^{s'}(\mathbb{R}, L^p(\mathbb{R}^m))}$ for a constant $c \ge 0$. Here we set $T *_+ f(t) = \int_0^t T(t-s)f(s)\,\mathrm{d}s$.