Nichtlineare Evolutionsgleichungen

10. Übungsblatt

Problem 1
Let \( m = 3 \) and \( \phi = (\phi_1, \phi_2) \in C^3(\mathbb{R}^2, \mathbb{R}^2) \) with \( \phi(0) = 0 \).

For \( u : \mathbb{R}^3 \to \mathbb{C} \) set \( F(u) = \phi_1(\bar{u}, 3u) + i\phi_2(\bar{u}, 3u) =: \phi(u) \). Show that \( F : H^2(\mathbb{R}^3) \to H^2(\mathbb{R}^3) \) is Lipschitz on bounded subsets of \( X = H^2(\mathbb{R}^3) \). Solve the nonlinear Schrödinger equation
\[
u'(t) = i\Delta u(t) + iF(u(t)), \quad t \geq 0, \quad u(0) = u_0,
\]
on \( X \) using the results of the first chapter of the lectures. Compare with Theorem 4.16 of the lectures (or Theorem 12.6 in the Internet Seminar Lecture Notes).

Problem 2
Let \( m \leq 5 \), \( \alpha \in (2, \alpha_c) \) and \( \mu \in \{-1, 1\} \). Set \( \phi(z) = z|z|^{\alpha - 1} \) for \( z \in \mathbb{R}^2 \). For \( b > 0 \) we write
\[
Z_1(b) = L^\infty([-b, b], H^1(\mathbb{R}^m)) \cap L^p([-b, b], W^{1,q}(\mathbb{R}^m))
\]
and
\[
\|u\|_{1,b} = \max \left\{ \|u\|_{L^\infty([-b, b], H^1(\mathbb{R}^m))}, \|u\|_{L^p([-b, b], W^{1,q}(\mathbb{R}^m))} \right\}, \quad u \in Z_1(b).
\]
We furthermore put \( F(u) = -i\mu|u|^{\alpha - 1}u \) for \( u \in Z_1(b) \).

(a) Show that \( \phi \in C^2(\mathbb{R}^2, \mathbb{R}^2) \) with \( \|\phi''(z)\| \leq c_0|z|^{\alpha - 2} \) for all \( z \in \mathbb{R}^2 \) and some constant \( c_0 > 0 \).

(b) Let \( b > 0 \) and \( u, v \in Z_1(b) \) with \( \|u\|_{1,b}, \|v\|_{1,b} \leq r \) for some \( r > 0 \). Let \( q = 1 + \alpha \) and \( \frac{q}{p} + \frac{\alpha}{q} = \frac{p}{2} \). Show that
\[
\|F(u) - F(v)\|_{L^p([-b, b], W^{1,q}(\mathbb{R}^m))} \leq c_\alpha b^{\frac{1}{q} - \frac{1}{p}} \|u - v\|_{1,b}
\]
for some constant \( c > 0 \) only depending on \( c_0, \alpha \) and \( m \).

(c) Let \( u_0 \in H^1(\mathbb{R}^m) \) and consider the nonlinear Schrödinger equation
\begin{equation}
u'(t) = i\Delta u(t) - i\mu|u|^{\alpha - 1}u, \quad t \in J, \quad u(0) = u_0.
\end{equation}
Show that there is a radius \( \delta > 0 \) and a time \( b_0 > 0 \) such that \( [-b_0, b_0] \subseteq J(u_0) \) for all \( v_0 \in \overline{B}_{H^1}(u_0, \delta) \) and \( \overline{B}_{H^1}(u_0, \delta) \to Z_1(b_0) \), \( v_0 \mapsto u(\cdot ; v_0) \),
is Lipschitz continuous, where \( u(\cdot ; v_0) \) is solution to (1) with initial value \( v_0 \) on its maximal interval of existence \( J(v_0) \).

Problem 3
Let \( V, \Gamma \in W^{1,\infty}(\mathbb{R}^m) \) be real valued. Set \( F_1(u) = -iV u \) and \( F_2(u) = -i\Gamma u |u|^{\alpha - 1} \) for \( u \in H^1(\mathbb{R}^m) \), as well as \( F = F_1 + F_2 \). Let \( \rho > 0 \) and \( u_0 \in H^1(\mathbb{R}^m) \) with \( \|v_0\|_{1,2} \leq \rho \).
Show that there is a time \( b_0 > 0 \) such that the nonlinear Schrödinger equation
\[
u'(t) = i\Delta u(t) + F(u(t)), \quad t \in [-b_0, b_0], \quad u(0) = u_0,
\]
has an \( H^1 \)-solution.

Hint: One can use Theorem 4.10 of the lectures (or Theorem 11.6 from the Internet Seminar Lecture Notes) with \( (p, q) = (\infty, 2) \).