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# Nichtlineare Evolutionsgleichungen

## 10. Übungsblatt

**Problem 1**

Let $m = 3$ and $\phi = (\phi_1, \phi_2) \in C^3(\mathbb{R}^2, \mathbb{R}^2)$ with $\phi(0) = 0$.
For $u : \mathbb{R}^3 \to \mathbb{C}$ set $F(u) = \phi_1(\Re u, \Im u) + \mathrm{i}\phi_2(\Re u, \Im u) =: \phi(u)$. Show that $F : H^2(\mathbb{R}^3) \to H^2(\mathbb{R}^3)$ is Lipschitz on bounded subsets of $X = H^2(\mathbb{R}^3)$. Solve the nonlinear Schrödinger equation

$$u'(t) = \mathrm{i}\Delta u(t) + \mathrm{i}F(u(t)), \quad t \geq 0, \qquad u(0) = u_0,$$

on $X$ using the results of the first chapter of the lectures. Compare with Theorem 4.16 of the lectures (or Theorem 12.6 in the *Internet Seminar Lecture Notes*).

**Problem 2**

Let $m \leq 5$, $\alpha \in (2, \alpha_c)$ and $\mu \in \{-1, 1\}$. Set $\phi(z) = z|z|^{\alpha-1}$ for $z \in \mathbb{R}^2$. For $b > 0$ we write

$$\mathcal{Z}_1(b) = L^\infty([-b, b], H^1(\mathbb{R}^m)) \cap L^p([-b, b], W^{1,q}(\mathbb{R}^m))$$

and

$$|||u|||_{1,b} = \max\left\{\|u\|_{L^\infty([-b,b],H^1(\mathbb{R}^m))}, \|u\|_{L^p([-b,b],W^{1,q}(\mathbb{R}^m))}\right\}, \quad u \in \mathcal{Z}_1(b).$$

We furhtermore put $F(u) = -\imath\mu|u|^{\alpha-1}u$ for $u \in \mathcal{Z}_1(b)$.

(a) Show that $\phi \in C^2(\mathbb{R}^2, \mathbb{R}^2)$ with $|\phi''(z)| \leq c_0|z|^{\alpha-2}$ for all $z \in \mathbb{R}^2$ and some constant $c_0 > 0$.

(b) Let $b > 0$ and $u, v \in \mathcal{Z}_1(b)$ with $|||u|||_{1,b}, |||v|||_{1,b} \leq r$ for some $r > 0$. Let $q = 1 + \alpha$ and $\frac{2}{p} + \frac{m}{q} = \frac{m}{2}$. Show that

$$\|F(u) - F(v)\|_{L^{p'}([-b,b],W^{1,q'}(\mathbb{R}^m))} \leq cr^{\alpha-1}b^{\frac{1}{p'}-\frac{1}{p}} \, ||| \, u - v|||_{1,b}$$

for some constant $c > 0$ only depending on $c_0$, $\alpha$ and $m$.

(c) Let $u_0 \in H^1(\mathbb{R}^m)$ and consider the nonlinear Schrödinger equation

$$u'(t) = \mathrm{i}\Delta u(t) - \mathrm{i}\mu|u|^{\alpha-1}u, \quad t \in J, \qquad u(0) = u_0. \tag{1}$$

Show that there is a radius $\delta > 0$ and a time $b_0 > 0$ such that $[-b_0, b_0] \subseteq J(v_0)$ for all $v_0 \in \overline{B}_{H^1}(u_0, \delta)$ and

$$\overline{B}_{H^1}(u_0, \delta) \to \mathcal{Z}_1(b_0), \quad v_0 \mapsto u(\,\cdot\,; v_0),$$

is Lipschitz continuous, where $u(\,\cdot\,; v_0)$ is solution to (1) with initial value $v_0$ on its maximal interval of existence $J(v_0)$.

**Problem 3**

Let $V, \Gamma \in W^{1,\infty}(\mathbb{R}^m)$ be real valued. Set $F_1(u) = -\mathrm{i}Vu$ and $F_2(u) = -\mathrm{i}\Gamma u|u|^{\alpha-1}$ for $u \in H^1(\mathbb{R}^m)$, as well as $F = F_1 + F_2$. Let $\rho > 0$ and $u_0 \in H^1(\mathbb{R}^m)$ with $\|u_0\|_{1,2} \leq \rho$. Show that there is a time $b_0 > 0$ such that the nonlinear Schrödinger equation

$$u'(t) = \mathrm{i}\Delta u(t) + F(u(t)), \quad t \in [-b_0, b_0], \qquad u(0) = u_0,$$

has an $H^1$–solution.

Hint: One can use Theorem 4.10 of the lectures (or Theorem 11.6 from the *Internet Seminar Lecture Notes*) with $(\overline{p}, \overline{q}) = (\infty, 2)$.