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Nichtlineare Evolutionsgleichungen

5. Übungsblatt

Problem 1

Assume A generates an analytic C_0-semigroup $T(\,\cdot\,)$ on X. Let $\alpha \in (0,1)$, $f \in C^\alpha([0,T],X)$ and $x \in D(A)$. Let u be the unique classical solution of

$$u'(t) = Au(t) + f(t), \qquad t \in [0,T], \qquad u(0) = x. \tag{1}$$

Show that, if $Ax + f(0) \in D_A(\alpha)$, then $u', Au \in C^\alpha([0,T],X)$ and $u' \in B([0,T], D_A(\alpha))$ with

$$\begin{aligned} &\|u\|_{C^{1+\alpha}([0,T],X)} + \|Au\|_{C^\alpha([0,T],X)} + \|u'\|_{B([0,T],D_A(\alpha)} \\ &\lesssim \|f\|_{C^\alpha([0,T],X)} + \|x\|_{D(A)} + \|Ax + f(0)\|_{D_A(\alpha)}. \end{aligned}$$

Hint: consider the splitting $u = u_1 + u_2$, where

$$\begin{aligned} u_1(t) &= \int_0^t T(t-s)(f(s) - f(t))\,\mathrm{d}s, \\ u_2(t) &= T(t)x + \int_0^t T(t-s)f(t)\,\mathrm{d}s. \end{aligned}$$

Problem 2

Assume A generates an analytic C_0-semigroup $T(\,\cdot\,)$ on X. Let $\alpha \in (0,1)$, $x \in D(A)$ and $f \in C([0,T],X) \cap B([0,T], D_A(\alpha))$. Let u be the unique classical solution of (1). Show that, if $Ax \in D_A(\alpha)$, then $u', Au \in C([0,T],X) \cap B([0,T], D_A(\alpha))$ and $Au \in C^\alpha([0,T],X)$ with

$$\begin{aligned} &\|u'\|_{B([0,T],D_A(\alpha))} + \|Au\|_{B([0,T],D_A(\alpha))} + \|Au\|_{C^\alpha([0,T],X)} \\ &\lesssim \|f\|_{B([0,T],D_A(\alpha)} + \|x\|_{D(A)} + \|Ax\|_{D_A(\alpha)}. \end{aligned}$$

Problem 3

Let Ω be a locally compact space and $m : \Omega \to \mathbb{C}$ a continuous function satisfying $\sup_{x\in\Omega} \Re(m(x)) < \infty$. Let $X = C_0(\Omega)$ and $T(\,\cdot\,)$ the C_0-semigroup on X generated by the multiplication operator

$$D(A) = \{u \in X : mu \in X\}, \quad Au = mu.$$

Show that, for each $\alpha \in (0,1)$,

$$D_A(\alpha, \infty) = \{u \in X : |m|^\alpha u \in C_b(\Omega)\}$$

and

$$D_A(\alpha) = \{u \in X : |m|^\alpha u \in X\}.$$

Beweis. Choose $C \in [0,\infty)$ such that $\sup_{x\in\Omega} \Re(m(x)) \le C$. Set $c := \min_{|z|=1} |e^z - 1| > 0$.

Note that $[T(t)f](x) = e^{tm(x)}f(x)$ for every $t \geq 0$ and $x \in \Omega$. So

$$\sup_{t\in(0,1)} t^{-\alpha}\|T(t)f - f\|_X = \sup_{t\in(0,1)} \sup_{x\in\Omega} t^{-\alpha}|(e^{tm(x)} - 1)f(x)|$$
$$= \sup_{x\in\Omega} \sup_{t\in(0,1)} t^{-\alpha}|(e^{tm(x)} - 1)f(x)|. \tag{2}$$

Let $f \in \{u \in X : |m|^\alpha u \in C_b(\Omega)\}$. Then have

$$\begin{aligned} t^{-\alpha}|(e^{tm(x)} - 1)f(x)| &= \left|\frac{e^{tm(x)} - 1}{t}\right|^\alpha |e^{tm(x)} - 1|^{1-\alpha}|f(x)| \\ &= \left|\frac{1}{t}\int_0^t m(x)e^{sm(x)}\,\mathrm{d}s\right|^\alpha |e^{tm(x)} - 1|^{1-\alpha}|f(x)| \\ &\leq e^{C\alpha}(e^C + 1)^{1-\alpha}|m(x)|^\alpha|f(x)| \\ &\leq e^{C\alpha}(e^C + 1)^{1-\alpha}\||m|^\alpha f\|_\infty. \end{aligned}$$

Therefore,

$$\sup_{t\in(0,1)} t^{-\alpha}\|T(t)f - f\|_X \overset{(2)}{\leq} e^{c\alpha}(e^c + 1)^{1-\alpha}\||m|^\alpha f\|_\infty,$$

showing that $f \in D_A(\alpha, \infty)$.

Conversely, let $f \in D_A(\alpha, \infty)$. Then,

$$\begin{aligned} |m(x)|^\alpha|f(x)| &\leq 1_{|m|\leq 2}2^\alpha\|f\|_\infty + 1_{|m|>2}|m(x)|^\alpha|f(x)| \\ &\leq 2^\alpha\|f\|_\infty + 1_{|m|>2}C^{-1}(1/|m(x)|)^{-\alpha}|(e^{(1/|m(x)|)m(x)} - 1)f(x)| \\ &\overset{(2)}{\leq} 2^\alpha\|f\|_\infty + C^{-1}\sup_{t\in(0,1)} t^{-\alpha}\|T(t)f - f\|_X, \end{aligned}$$

showing that $f \in \{u \in X : |m|^\alpha u \in C_b(\Omega)\}$.

To finish the proof, we show that $\{u \in X : |m|^\alpha u \in X\}$ equals the closure of $D(A)$ in $\{u \in X : |m|^\alpha u \in C_b(\Omega)\}$. Note that $D(A) \subset \{u \in X : |m|^\alpha u \in X\}$ as $|m|^\alpha \leq 1 + |m|$. Using that X is closed in $C_b(\Omega)$, we find that $\{u \in X : |m|^\alpha u \in X\}$ is closed in $\{u \in X : |m|^\alpha u \in C_b(\Omega)\}$. It follows that the closure of $D(A)$ in $\{u \in X : |m|^\alpha u \in C_b(\Omega)\}$ is contained in $\{u \in X : |m|^\alpha u \in X\}$. For the reverse inclusion, let $f \in \{u \in X : |m|^\alpha u \in X\}$. Let $\epsilon > 0$. Choose a compact K in Ω such that $|f(x)| < \epsilon$ and $||m(x)|^\alpha f(x)| < \epsilon$ for every $x \in \Omega \setminus K$. Pick $\chi \in C_c(\Omega)$ with $\chi = 1$ on K. Then $\tilde{f} := \chi f \in D(A)$ with

$$\|f - \tilde{f}\|_\infty < \epsilon, \qquad \||m|^\alpha(f - \tilde{f})\|_\infty < \epsilon.$$

This completes the proof. $\square$