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1^{st} exercise sheet
PARTIAL DIFFERENTIAL EQUATIONS
WS 2006/07

1. Let $v : \mathbb{R} \to \mathbb{R}$ smooth and $u(t,x) := v(\frac{x^2}{t})$.

   (i) Show that for all $x \in \mathbb{R}$ and $t \in \mathbb{R} \setminus \{0\}$ holds $u_t = u_{xx}$ if and only if for all $z > 0$ holds
$$4zv''(z) + (2+z)v'(z) = 0. \tag{1}$$

   (ii) Show that the general solution of (1) is
$$v(z) = c \int_0^z e^{-s/4} s^{-1/2} \, ds + d.$$

2. Let $\varphi : \mathbb{R} \to (0,\infty)$ be continuous and let $v : \mathbb{R} \to \mathbb{R}$ be a C^1-function. Solve
$$\begin{aligned} &u_t(t,x) = \varphi(x) u_x(t,x), \quad t \geq 0, x \in \mathbb{R}, \\ &u(0,\cdot) = v. \end{aligned} \tag{2}$$

   Does the solution exist globally?

   Hint: Suppose, for x_0 fixed, that a function $g : [0,\infty) \to \mathbb{R}$ exists such that $g(0) = x_0$ and $t \mapsto u(t, g(t))$ is constant.

3. Let $u : \mathbb{R}^4 \to \mathbb{R}$ a solution of the heat equation
$$u_t = \Delta u.$$
   Further let $A \in \mathbb{R}^{3,3}$ unitary (this means $A^T = A^{-1}$) and $v(t,x) := u(t, Ax)$. State a simple second order differential equation, of which v is a solution.

4. (i) Let $p :\in C^1(\mathbb{R}^{n+1}, \mathbb{R})$ and $u : \mathbb{R}^{n+1} \to \mathbb{R}^n$ a smooth solution of
$$u_t - \Delta u + (u \cdot \nabla) u + \nabla p = 0.$$
   Further let $q(t,x) := \lambda^2 p(\lambda^2 t, \lambda x)$ for some $\lambda \in \mathbb{R}$. Find a solution of
$$v_t - \Delta v + (v \cdot \nabla) v + \nabla q = 0.$$

   (ii) Let $u : (0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ a smooth solution of the heat equation $u_t = \Delta u$. For $\lambda \in \mathbb{C}$ also $u_\lambda(t,x) := u(\lambda^2 t, \lambda x)$ solves the heat equation. Use this to show, that $w : (0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ solves the heat equation as well, with
$$w(t,x) := x \cdot \nabla u(t,x) + 2t u_t(t,x).$$