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10^{th} exercise sheet
PARTIAL DIFFERENTIAL EQUATIONS
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37. For $(x, y) \in \mathbb{R}^2$ we introduce the polar coordinates $\varphi = \varphi(x, y) := \arctan \frac{y}{x}$ and $r = r(x, y) := (x^2 + y^2)^{1/2}$.

    Let $\Phi \in (0, 2\pi)$. For $u \in C^2(\{(r\cos\varphi, r\sin\varphi)^T : r \in [0, 1], \varphi \in [0, \Phi]\})$ we define $v(r, \varphi) := u((r\cos\varphi, r\sin\varphi)^T)$.

    Show that the Laplace operator in polar coordinates reads:

$$\Delta u(x, y) = \partial_r^2 v(r, \varphi) + \frac{1}{r}\partial_r v(r, \varphi) + \frac{1}{r^2}\partial_\varphi^2 v(r, \varphi).$$

38. Let $\Omega \subset \mathbb{R}^2$ open and $\xi \in \partial\Omega$ such the following cone-property holds:

$$\{\xi + (r\cos\varphi, r\sin\varphi) : r \in [0, 1], \varphi \in [\Phi, \Phi_2]\} \cap \Omega = \{\xi\}.$$

    For some $\Phi < \Phi_2$. Show that ξ is regular.

    Hint: Assume w.l.o.g. $\xi = 0$, $\Phi \in (0, 2\pi)$ and $\Phi_2 = 2\pi$.
    Use problem 37 to find a barrier function of the shape $r^\lambda f(\varphi)$ for $r \in [0, 1]$, $\varphi \in [0, \Phi]$, which is harmonic in the interior and zero for $\varphi \in \{0, \Phi\}$.

39. Let u,v and w non-constant C^2-solutions of

$$\begin{aligned}
&u_{xy}(x, y) = 0,\\
&v_{xx} + 2v_{xy}(x, y) + 2v_{yy}(x, y) + \sin(x)v_y(x, y) = y^2,\\
&\cosh(y^2)w_{xx}(x, y) + \big(\sinh(x^2)\sinh(y^2)\big)^{1/2} w_{xy}(x, y) + \cosh(x^2)w_{yy}(x, y) = 1
\end{aligned}$$

    in $\mathbb{R}^2$. May u,v or w have local maxima?

40. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $\lambda > 0$ and $f \in C^0(\Omega)$.
    Suppose that $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfies $u - \lambda\Delta u = f$ in Ω and $u = 0$ on $\partial\Omega$.

    Show $f \geq 0 \Longrightarrow u \geq 0$, $f \leq 1 \Longrightarrow u \leq 1$ and $|f| \leq 1 \Longrightarrow |u| \leq 1$.

    Hint: Take a look at the proof of the maximum principle in § 8.