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11^{th} exercise sheet
PARTIAL DIFFERENTIAL EQUATIONS
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41. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ such that

$$\begin{aligned} \Delta u &= u^3 \quad \text{in } \Omega, \\ u|_{\partial\Omega} &= 0. \end{aligned}$$

Show $u = 0$ in Ω.

42. Let $\Omega \subseteq \mathbb{R}^n$ be a domain and $u \in C^2(\Omega)$ a solution of the Monge-Ampère equation

$$\det\left((\partial_j \partial_k u(x))_{jk}\right) = f(x),$$

where $f > 0$ on Ω. Assume there is $x_0 \in \Omega$ such that the matrix $(\partial_j \partial_k u(x_0))_{jk}$ is positive definite. Show that the equation is elliptic at u and every $x \in \Omega$.

43. Let

$$u_1(x,y) := \frac{1}{2} + \frac{1}{2}(x^2 + y^2),$$

$$u_2(x,y) := \frac{3}{2} - \frac{1}{2}(x^2 + y^2), \quad x, y \in \mathbb{R}.$$

Show that u_1 and u_2 solve the Monge-Ampère-equation

$$u_{xx}u_{yy} - u_{xy}^2 = 1$$

and that

$$u_1|_{\partial B(0,1)} = u_2|_{\partial B(0,1)} = 1.$$

Is this a contradiction to the results on nonlinear elliptic equations?

44. Let $x \in \mathbb{R}^n$, $R > 0$ and $u \in C^2(B(x,R))$. Let $a = (a_{jk})_{jk}$ be a symmetric and positive semidefinite matrix, and denote by $a^{1/2}$ the square root of a, i.e. the unique symmetric and positive semidefinite matrix b such that $b^2 = a$. Show that

$$\sum_{j,k=1}^{n} a_{jk} \partial_j \partial_k u(x) = \lim_{h \to 0} c_n \int_{B(0,1)} \frac{u(x + ha^{1/2}y) - u(x)}{h^2} \, dy,$$

where $c_n := 2\left(\int_{B(0,1)} y_1^2 \, dy\right)^{-1} = \frac{2(n+2)}{\mathrm{vol}_n(B(0,1))}$.

Hint: Taylor expansion.