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12^{th} exercise sheet
PARTIAL DIFFERENTIAL EQUATIONS
WS 2006/07

45. Let $\Omega \subset \mathbb{R}^n$ bounded, $\Omega_T := \Omega \times (0, T)$ and

$$L := \sum_{j,k=1}^{n} a_{jk}(x,t) \frac{\partial^2}{\partial x_j \partial x_k} + \sum_{j=1}^{n} b_j(x,t) \frac{\partial}{\partial x_j}$$

be elliptic for all $(x,t) \in \Omega_T$.
Show: For a function $u \in \mathbb{C}^0(\overline{\Omega}_T)$ which is twice differentiable with respect to $x \in \Omega$ and once with respect to $t \in (0,T)$ holds

$$u_t \leq Lu \qquad \Longrightarrow \qquad \sup_{\Omega_T} u = \sup_{\partial^* \Omega_T} u.$$

46. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $f \in C_b(\Omega)$. Define

$$u(x) := \int_\Omega \Gamma(x-y) f(y)\, dy, \quad x \in \mathbb{R}^n,$$

where Γ is the fundamental solution for the Laplacian on $\mathbb{R}^n$. We know that $\Delta u = f$ in Ω in the sense of distributions. For $h > 0$ and $x \in \Omega$ we define

$$\Delta_h u(x) := c_n \int_{B(0,1)} \frac{u(x+h\xi) - u(x)}{h2}\, d\xi,$$

where $c_n := (\int_{B(0,1)} y_1^2\, dy)^{-1}$. This exercise will show that, for fixed $x \in \Omega$, we have

$$\lim_{h \to 0+} \Delta_h u(x) = f(x),$$

The limit on the left hand side may be taken as an extension of Δ (cp. Exercise 44) for the case that $u \notin C^2(\Omega)$.
For $z \in \mathbb{R}^n$, let $\rho_h(z) := c_n \int_{B(0,1)} \frac{\Gamma(z+h\xi)-\Gamma(z)}{h^2}\, d\xi$. We fix $x \in \Omega$.

(a) Show that $\Delta_h u(x) = \rho_h * f_0(x)$ where f_0 denotes the zero-extension of f to $\mathbb{R}^n$.

(b) Show that $\rho_h = h^{-n} \rho(\frac{\cdot}{h})$ where $\rho := \rho_1$. (Hint: We know Γ explicitly.)

(c) Show that $\rho(z) = 0$ for $|z| > 1$. (Hint: mean value property.)

(d) Show that $\rho \in L^1(\mathbb{R}^n)$.

(e) Show that $\int_{\mathbb{R}^n} \rho(z)\, dz = 1$. (Hint: Show $\int \rho_h(z)\, dz \to 1$ $(h \to 0+)$.)

(f) Conclude $\Delta_h u(x) \to f(x)$ as $h \to 0+$.

Remark: end of problem sheet.