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13^{th} exercise sheet
PARTIAL DIFFERENTIAL EQUATIONS
WS 2006/07

47. Let $f \in C_b^0(\mathbb{R}^n)$ and $u : \mathbb{R}^n \times [0, \infty)$ the solution of the heat equation with $u(\cdot, 0) = f$.
a) Show that for $x, z \in \mathbb{R}^n$ holds
$$u(x,t) - u(z,t) \to 0.$$
b) Let further $f(x) \to v$ as $|x| \to \infty$. Show that
$$u(x,t) \to v.$$
c) What happens if $\operatorname{supp} f$ is bounded?

48. Let K the kernel (fundamental solution) of the heat equation ($K(x,t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$).
Show that for $s, t > 0$ holds
$$K(x, t+s) = \int_{\mathbb{R}^n} K(x-y, t) K(y, s) dy.$$
Hint: Uniqueness of the solution of the heat equation.

49. Let $k, h > 0$ and $\Sigma := (nh, mk) : n, m \in \mathbb{Z}, m \geq 0$. Let v a solution of the discrete heat equation
$$\frac{v(x, t+k) - v(x,t)}{k} - \frac{v(x+h, t) - 2v(x,t) + v(x-h, t)}{h^2} = 0 \text{ on } \Sigma$$
with $v(x, 0) = f(x) \in \mathbb{C}^0(\mathbb{R})$.
Show that for the case $\frac{k}{h^2} = \frac{1}{2}$ holds
$$v(nh, mk) = 2^{-m} \sum_{j=0}^{m} \binom{m}{j} f\big((n - m + 2j)h\big),$$
and therewith $\sup_{\Sigma} |v| \leq \sup_{\mathbb{R}} |f|$.

50. Let $\mathbf{\Gamma}(x) := \int_0^\infty e^{-t} t^{x-1} dt$. Show:
$$\mathbf{\Gamma}(x+1) = x\mathbf{\Gamma}(x), \qquad (x \geq 0), \qquad (1)$$
$$\mathbf{\Gamma}(1) = 1, \qquad (2)$$
$$\int_0^\infty s^n e^{-s^2} ds = \frac{1}{2}\mathbf{\Gamma}\Big(\frac{n+1}{2}\Big), \qquad (n \in \mathbb{N}), \qquad (3)$$
$$\mathbf{\Gamma}\Big(\frac{1}{2}\Big) = 2\int_0^\infty e^{-s^2} ds = \sqrt{\pi}, \qquad \text{(the second equation is known)}, \qquad (4)$$
$$\sqrt{\pi}^n = \int_{\mathbb{R}^n} e^{-|x|^2} dx = \frac{1}{2}\mathbf{\Gamma}\Big(\frac{n}{2}\Big) \operatorname{vol}(S^{(n-1)}), \qquad \text{(volume of the } n-1 \text{ sphere)}. \qquad (5)$$
Use this to fianlly conclude the missing statement from the lecture
$$\frac{1}{2} \operatorname{vol}(S^{(n-2)}) \pi^{-\frac{n}{2}} \int_0^\infty \sigma^{-\frac{1}{2}} e^{-\sigma} \Big(\int_0^\infty e^{-s^2} s^{n-2} ds \Big) \, d\sigma = \frac{1}{2}.$$