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14^{th} exercise sheet
PARTIAL DIFFERENTIAL EQUATIONS
WS 2006/07

51. Let u a C^2-solution of the wave-equation with initial conditions $u_t(x,0) = 0$ and $u(x,0) = f(x)$ for $x \in \mathbb{R}^n$, where f has compact support and n is odd. Show that
$$v(x,t) := \int_{-\infty}^{\infty} \frac{e^{-\frac{s^2}{4t}}}{\sqrt{4\pi t}} u(x,s)ds$$
solves the heat equation $v_t = \Delta_x v$ for $t > 0$ with initial condition $v(\cdot,0) = u(\cdot,0) = f$.
Remark: Do not care, why you may change the order of integration and differentiation.

52. We define the waek solution of the one-dimensional wave equation to be a function u such that
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x,t)\big(\Phi_{tt}(x,t) - \Phi_{xx}(x,t)\big)\ dx\ dt = 0$$
For all $\Phi \in C_c^2(\mathbb{R}^2)$.
a) Show that any solution of the wave-equation is a weak solution.
b) Show that the discontinuos functions $u(x,t) := H(x-t)$ and $v(x,t) := H(x+t)$ are weak solutions, where the Heaviside function H is the characteristic function of $[0,\infty)$.
c) Try to give a usefull definition for weak solutions of the wave-equation in $\mathbb{R}^n$.

53. Let $f, g : [0,\infty) \to \mathbb{R}$ continuous. Find solutions u and v of the one-dimensional wave-equation
$$\begin{array}{ll} u(x,0) = v(x,0) = f(x), & \text{for } x \in [0,\infty), \\ u_t(x,0) = v_t(x,0) = g(x) & \text{for } x \in [0,\infty), \\ u(0,t) = v_x(0,t) = 0 & \text{for } t \in [0,\infty). \end{array}$$

54. Let u a solution of the wave-equation in $\mathbb{R}^5$ with initial conditions $u_t(x,0) = g(x)$ and $u(x,0) = f(x)$ for all $x \in \mathbb{R}^5$. We define
$$N(x,r,t) := r^2 \frac{\partial}{\partial r} S_r\big(u(\cdot,t)\big)(x) + 3rS_r\big(u(\cdot,t)\big)(x)$$
a) Show that N is a solution of
$$N_{tt} = N_{rr}$$
and find N from its initial data in terms of $S_r f$ and $S_r g$.
b) Show that
$$\begin{aligned} u(x,t) &= \lim_{r\to 0} \frac{N(x,r,t)}{3r} \\ &= \Big(\frac{1}{3}t^2\frac{\partial}{\partial t} + t\Big)S_t g(x) + \frac{\partial}{\partial t}\Big(\frac{1}{3}t^2\frac{\partial}{\partial t} + t\Big)S_t f(x) \end{aligned}$$