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# $2^{nd}$ exercise sheet
PARTIAL DIFFERENTIAL EQUATIONS
WS 2006/07

5. Proof the uniqueness of a $C^1$-solution of the wave equation.
   This means for $f, g \in C^2(\mathbb{R})$ there exists exactly one function $u \in \mathbb{C}^1$ with
$$\begin{aligned} u_{tt} - u_{xx} &= 0 && \text{in } (0,\infty)\times\mathbb{R}, \\ u(0,\cdot) = g, \quad u_t(0,\cdot) &= h && \text{on } \mathbb{R}. \end{aligned} \tag{1}$$

6. Let $f, g \in C^2([0,\infty))$. Try to solve
$$\begin{aligned} u_{tt} - u_{xx} &= 0 && \text{in } (0,\infty)\times(0,\infty), \\ u(0,\cdot) = g, \quad u_t(0,\cdot) &= h && \text{on } [0,\infty), \\ u(t,0) &= 0 && \text{for all } t \in [0,\infty). \end{aligned} \tag{2}$$
   Do we need any further assumptions on $f$ and $g$?
   Hint: Try to extend $f$ and $g$ to functions in $C^2(\mathbb{R})$ and solve the wave equation.

7. Let $g \in L^1(\mathbb{R}^n, \mathbb{R})$ with $\int_{\mathbb{R}^n} g(x)dx = 1$. For $\varepsilon > 0$, we define $g_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ by
$$g_\varepsilon(x) := \varepsilon^{-n} g\big(\frac{x}{\varepsilon}\big), \quad x \in \mathbb{R}^n.$$
   For a bounded and uniformly continuous function $f : \mathbb{R}^n \to \mathbb{R}$ we set
$$f_\varepsilon(x) := \int_{\mathbb{R}^n} g_\varepsilon(x-y) f(y)\, dy.$$
   Show that $f_\varepsilon$ tends to $f$ uniformly ($\varepsilon \to 0^+$).

8. Let $(g_k)$ a sequence in $L^1(\mathbb{R}^n)$ with
$$\begin{aligned} &\int_{\mathbb{R}^n} g_k(x)dx = 1 \text{ for all } k \in \mathbb{N} && (D1), \\ &\sup\{||g_k||_1 : \ k \in \mathbb{N}\} < \infty && (D2), \\ &\forall \delta > 0 : \lim_{k\to\infty} \int_{|x|>\delta} |g_k(x)|dx = 0 && (D3). \end{aligned}$$
   Further let $y \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ a bounded and measurable function which is continuous at $y$. Show the convergence
$$\int_{\mathbb{R}^n} g_k(y-x) f(x)\, dx \longrightarrow f(x) \ \ (k \to \infty).$$
   Remark: A sequence $(g_k)$ satisfying $(D1)$, $(D2)$ and $(D3)$ is called Dirac-sequence.