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4^{th} exercise sheet
PARTIAL DIFFERENTIAL EQUATIONS
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13. Let $(\mu_k)_{k\in\mathbb{N}} \in (0,\infty)^{\mathbb{N}}$ such that $\sum_{k=1}^{\infty} \frac{1}{\mu_k} < \infty$. Let $\varphi_k(t) := \mu_k e^{-\mu_k t}\, \mathbb{1}_{[0,\infty)}(t)$ for all $t \in \mathbb{R}, k \in \mathbb{N}$, and $\psi_{m,n} := \varphi_m * \varphi_{m+1} * \cdots * \varphi_n$ for $m, n \in \mathbb{N}$ with $m \geq n$.

(i) Show

$$\|\psi_{m,n}\|_{L_1} = \int_0^\infty \psi_{m,n}(t)\, dt = 1 \text{ and } \int_{-\infty}^\infty |t|\psi_{m,n}(t) = \sum_{k=m}^n \frac{1}{\mu_k}$$

for all $m, n \in \mathbb{N}$ with $m \geq n$.

Hint: Induction on n for fixed m.

(ii) Show that $\psi_{1,n} \in C^{n-2}$ for $n \in \mathbb{N}_{\geq 2}$ and

$$\|\psi_{1,n}^{(j)}\|_\infty \leq 2^j \prod_{k=1}^{j+1} \mu_k =: \lambda_{j+1} \quad \text{for } j \in \mathbb{N}_0, n \in \mathbb{N}_{\geq j+2}.$$

Hint: Use exercise 12 and $\|g * f\|_\infty \leq \|g\|_{L_1} \|f\|_{L_\infty}$.

(iii) Show that $\left(\psi_{1,m}^{(j)}\right)_{m\geq j+2}$ converges uniformly on $\mathbb{R}$.

Hint: Write

$$\psi_{1,m}^{(j)} - \psi_{1,n}^{(j)} = \psi_{1,m}^{(j)} - \psi_{1,m}^{(j)} * \psi_{m+1,n} \qquad (*)$$

and use (ii) for $j+1$.

(iv) Show that $(\psi_{1,m})_{m\in\mathbb{N}}$ is a $\|\cdot\|_{L_1}$-Cauchy-sequence.

Hint: Use $(*)$ for $j = 0$, look at the proof of Theorem 2.6, observe that (i) implies $\int_\delta^\infty \psi_{m+1,n}(s)\, ds \leq \frac{1}{\delta} \sum_{k=m+1}^n \frac{1}{\mu_k}$ for all $\delta > 0$ and use $\|\psi_{1,m}'\|_{L_1} \leq 2$.

14. Consider the backward heat equation

$$\begin{aligned} u_t + \Delta u &= 0 \quad \text{on } \mathbb{R}^n, t > 0, \\ u(0,\cdot) &= f \quad \text{on } \mathbb{R}^n, \end{aligned} \qquad (1)$$

where $f \in C_b(\mathbb{R}^n)$. Show instability of this equation, that is, show that for all $\varepsilon, M, t_0 \in \mathbb{R}_{>0}$ exists an $f \in C_b(\mathbb{R}^n)$ and a separation-of-variables solution u of (1) such that $\|f\|_\infty \leq \varepsilon$ and $\|u(t_0,\cdot)\|_\infty \geq M$.

Hint: Consider first the case $n = 1$. For the general case search for eigenfunctions of Δ with separated variables.

15. Consider the first order PDE

$$F(\nabla u(x), u(x), x) = 0 \text{ in } \Omega. \tag{2}$$

The PDE (2) is called linear, if it is given by

$$0 = F(\nabla u(x), u(x), x) = a(x) \cdot \nabla u(x) + b(x)u(x) + c(x),$$

where $a(x) \in \mathbb{R}^n, b(x), c(x) \in \mathbb{R}$ for all $x \in \Omega$. Derive the characteristic ODE in this case.

16. (i) Consider the PDE

$$\begin{aligned} &x_1\partial_1 u + 2x_2\partial_2 u + \partial_3 u = 3u \quad \text{on } \mathbb{R}^3, \\ &u|_\Gamma = g, \end{aligned}$$

where $\Gamma := \{x \in \mathbb{R}^3 \,|\, x_3 = 0\}$. Solve this equation with the method of characteristics.

(ii) Consider the PDE

$$\begin{aligned} &\partial_1 u + \partial_2 u = u^2 \quad \text{on } \Omega, \\ &u|_\Gamma = g, \end{aligned}$$

where $\Omega := \{x \in \mathbb{R}^2 \,|\, x_2 > 0\}$ and $\Gamma := \partial\Omega := \{x \in \mathbb{R}^2 \,|\, x_2 = 0\}$. Solve this equation with the method of characteristics.