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6^{th} exercise sheet
PARTIAL DIFFERENTIAL EQUATIONS
WS 2006/07

21. Solve the fully nonlinear problem

$$\begin{aligned} u_x u_y &= u \quad \text{in } U \\ u(x,y) &= y^2 \quad \text{on } \Gamma, \end{aligned} \tag{1}$$

where $U := \{(x,y) \in \mathbb{R}^2 : x > 0\}$ and $\Gamma := \{(0,y) : y \in \mathbb{R}\}$.

22. Let $x \in \mathbb{R}^n$ with $|x_j| < 1$ for $j \le n$. Show that

$$\sum_\alpha x^\alpha = \prod_{j=1}^n \frac{1}{(1-x_j)} = \frac{1}{(1-x_1)(1-x_2)\cdots(1-x_n)}.$$

Set $x := (q, q, \ldots, q)$ and apply D^β to both sides to show

$$\sum_{\alpha \ge \beta} \frac{\alpha!}{(\alpha-\beta)!} q^{|\alpha-\beta|} = \frac{\beta!}{(1-q)^{n+|\beta|}}.$$

23. Show that the function

$$f(x) := \sum_{n=1}^\infty \frac{\cos(n!x)}{(n!)^n}$$

is of class C^∞ on $\mathbb{R}$, but is not real analytic anywhere.

Hint: Show first that f is not in $C_{M,r}(0)$ for any M and r.
Next show, that $f(x) - f(x + 2\pi q)$ is analytic for any rational number q.

24. Let f be real analytic in a convex domain D which contains the origin. Let

$$\sum_\alpha c_\alpha x^\alpha$$

be the associated Tayler series in the origin. Assume this series converges absolutely for some $x \in D$. Show that it converges to $f(x)$.

Hint: Consider $f(\lambda x)$ for $0 < \lambda < 1$ and pass to the limit $\lambda \to 1$.